[Academic Script]

Number System & Set Theory

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Academic Script

1. Introduction

The importance of Mathematics in Business Economics can be highlighted by the simple fact that almost all persons who have got Nobel Prize in Economics had very strong background in Mathematics. Actually John Nash one of the pioneers of Game Theory who got Nobel Prize and on whose life the well-known Oscar Prize Winner film “Beautiful Mind” was made was a Mathematician. No work of importance draws the attention of people unless it uses sophisticated Mathematics and Statistics.

2. Number System

We all are aware of Whole Numbers which are also known as Natural Numbers or Positive Integers. The learning of counting

$$1, 2, 3, 4, \ldots$$

starts these days in the form of pre-nursery-home-education. These numbers are taken as self-evident, indefinable and having the property that they can be written down in succession one after the other without end. Also we wish to highlight the often unnoticed, ordinal i.e. related to order, aspect of the Natural Numbers, which is imbedded in the counting process and which is also intended to be noticed in the figure when we move on the right we get larger and larger numbers. Let us recall a Famous Quote of Leopold Kronecker, "God made the natural numbers; all else is the work of men"

The rigorous or logical development of the number system which is far from simple is beyond the scope of discussion here. However the story of progressive development of the Number System can be narrated. And this is what we propose to do now.

After the positive integers, one must familiarize oneself with the number zero, a number with the property that when it is added to any other number we get the same number. Note that zero was invented in India. For more information see Discovery of Zero-BBC India on youtube. It is a mystical number and its use without proper understanding may lead to wrong results. As for Example one should never divide any number by 0. It is prohibited in mathematics. Some of the false results that one can get, if one overlooks this advice, are nicely and delicately discussed in the literature.

We also have negative integers, one negative integer for each positive integer. In other words there are as many negative integers as are the positive integers. There are many uses of negative integers. One can easily see through examples the natural occurrence of negative numbers and also their use.

Example:

Suppose one is talking of profit and loss. And if one encounters the negative profit, it may be interpreted as loss. Likewise negative loss can be interpreted as profit.

In day to day life one has to consider the problems of division: e.g. distribute 100 objects into 20 equal parts and divide 2 apples among three persons so on…. So that, it is not difficult to imagine how important are the numbers of the type $\frac{p}{q}$**?** Here p is an integer and q is a positive integer. These numbers are known as fractions, they are also called the rational numbers. Their decimal representations are either terminating, repeating or have recurring cycle as we see in:

$$\frac{1}{2} = 0.5 = 0.4999\ldots$$

$$\frac{1}{3} = 0.333\ldots$$

$$\frac{1}{7} = 0.\overline{142857}\,\overline{142857}\ \ldots$$

Starting from the very early time of the development of the human civilization the problems involving right-angled triangles and circles came into discussions in varied situations. And they had to think of numbers which are denoted by $\sqrt{2}\ and\ \pi$ in many different situations; these are numbers that cannot be represented or put in the form $\frac{p}{q}$ **.** Such numbers are called irrational numbers. Their decimal representations are neither terminating nor have repeating or recurring cycle.

Another such number which comes up very frequently is denoted by *e*. Note that the approximate values denoted
by notation $\cong$, of these numbers are:

$$\sqrt{2} \cong 1.414213562$$

$$\pi \cong 3.141592654$$

$$e \cong 2.718281828$$

We should take care not to take these values as exact or equal they are only approximate values.

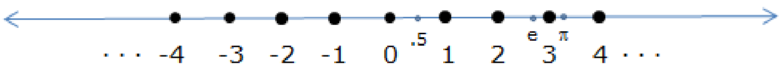
Also note that rational number 22/7; a value often taken as the value of π is only an approximate value of π which is correct only up to two decimal places after decimal point.

All these numbers including the earlier defined numbers are called real numbers. Thus progressively we have got familiarity with:

Natural Numbers, Integers, Rational Numbers and Real Numbers

<u>Real line:</u>

It is very convenient to represent each real number on a line. In fact we associate with every real number a unique point on the line and other way every point on the line can be associated with a unique real number. That is, we visualize the line, marked with all the real numbers as:



Square of any real number is non-negative i.e. it is either positive or zero. Thus square-roots of non-negative numbers only can be found. And if want to find square-roots of negative numbers we have to extend our number system still further from real numbers. This is done by writing and inventing the new notation i for the square-root of -1 i.e. $\sqrt{-1} = i$ or which is same as $i^2 = -1$ and then defining numbers $a + ib$ where a and b are real numbers. These numbers are called complex numbers. The sum and multiplication of complex numbers are defined in natural way, keeping in mind that $i^2 = -1$. Thus,

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Now we wind up the whole story of the number system by completing the hierarchy:

Natural Numbers, Integers, Rationals, Reals, Complex numbers.

Logic & Concept and Methods of Proof:

Though the so called useful mathematics is created out of the practical needs, its solid foundation rests on the logical, axiomatic development of the subject. The classic example is the Euclidean geometry that we all study in our schools.

The rigorous mathematical development usually consists of:

1. Some undefined terms
2. Axioms or postulates
3. Consistency and Independence of the system
4. Definitions
5. Examples
6. Lemmas
7. Theorems
8. Corollaries
9. Applications

Mathematical statements or simply statements have definite truth value. A statement thus is either true or false. Note that

Mathematical statements are eternal truths. They wear different look only when you change your system i.e. when you change your axioms and/or definitions.

One may also say that mathematical truths are simply logical deductions of your definitions and axioms.

The logic that we use has the following main aspects:

1. Negation of the statement p which is denoted by ~p
2. The implication that "p implies q" is equivalent to "~q implies ~p". The method of Reductio ad Absurdum which is also known as method of contradiction. In this method of proof one arrives at some absurd or untenable statement if one assumes that what is to be proved is false.

For further details about Logic one may refer to a book by R L Wilder entitled "Introduction to the foundations of Mathematics".

We in mathematics are often confronted with several statements based on the set of positive integers. Their validity can't be established or proved by simple verification of few cases. As you can see the following three statements are valid for a large number of initial values, however they are false statements.

1. Every positive integer n is less than 10,000.
2. The formula $f(n) = n^2 + n + 41$ gives a prime number for each positive integer n.

Why are these statements false?

1. Statement number one is false because if we take n =10,000 the statement is false. In fact it is false except for the initial 9,999 cases.
2. This interesting formula produces primes for initial 40 values of n i.e. for $n = 0$ to 39. But when you put $n = 40$, it gives a number $40^2+40+41=41^2$

which is not a prime number.

We conclude this discussion with a simple note that proof is a serious business in Mathematics. However, when we want to apply mathematics, we will not go too much into this part of mathematics here.

Equations and Inequalities:

In economic analysis and elsewhere one comes across different types of equations:

1. A definitional equation
2. A behavioral equation
3. Assertive equation
4. Equilibrium conditions

We shall elaborate each one, one after the other now:

1. A definitional equation:

Example:

$$\boldsymbol{\pi} = \boldsymbol{R} - \boldsymbol{C}$$

Here $\boldsymbol{R}$ is total Revenue $\boldsymbol{C}$ is total Cost. Now total Profit denoted here by notation π is by definition $R - C$. Here both the quantities are identically equal and the notation "$\equiv$" is also some time used. The modern notation for this definitional use of "=" is " $:=$ " or " $\stackrel{\text{def}}{=}$ ". Thus each of,

$$\boldsymbol{\pi} = \boldsymbol{R} - C$$

$\boldsymbol{\pi} \equiv \boldsymbol{R} - C$

$\boldsymbol{\pi} := \boldsymbol{R} - C$

$\boldsymbol{\pi} \stackrel{\text{def}}{=} \boldsymbol{R} - C$

conveys the same thing that Profit i.e. $\boldsymbol{\pi}$, is by definition equal to the excess of Revenue over Cost i.e. $\boldsymbol{R}$ over Cost.

2. A behavioral equation: When behavior of some variable, often called dependent variable, is described in terms of some other variable, which is usually called independent variable; we refer it as behavioral equation.

Example: The three cost functions:

a. $C = 80 + Q^2$

b. $C = 100 + Q$

c. $C = 100 + 25Q$

describe the Cost C in three different situations in terms of the quantity of output denoted by Q which is considered here as an independent variable. Note that one supplies different values for Q and Cost i.e. C calculated using the formulae under consideration.

3. Assertive equation:

Here in certain situation our variables are having certain value in them and then we say that the arithmetical expression has certain value:

Example: When we know in advance that x has got definite value say 1 i.e. $x = 1$ then

$$(x + 4)x(x - 4)(x - 3) = 30$$

4. Equilibrium conditions:

Suppose Quantity demanded is given by

$\boldsymbol{Q_d = a - bP}$

and Quantity supplied is given by

$$\boldsymbol{Q_s = -c + dP}$$

where P is the profit. In economics it is a standard assumption that market equilibrium takes place if and only if there is no excess demand i.e. $\boldsymbol{Q_d - Q_s = 0}$.

Thus equilibrium condition is

$$Q_d = Q_s$$

Where in we are supposed to find the value of P may be called $\bar{P}$ for which the corresponding values of Q_d and Q_s are equal say $\bar{Q}$. Note the different usages of the phrase "equation" and the appropriate meaning of the notation "=". There should not be any confusion when only the single notation "=" is used to mean so many different things. Only the context makes it clear as to what is meant in the definite situation. To avoid confusion the other notations may also be used for safe communication.

3. Set Theory

Note that set theory is an indispensable tool for understanding any serious part of mathematics or its applications. It is also called the language of modern mathematics.

A "set" is simply a well-defined collection of distinct objects. These objects may be anything.

Example:

1) The collection of four integers -4,0,4,3.
2) The collection of all the subjects offered in B.A. degree course of Delhi University.
3) The collection of all Integers.

All of the above collections are sets. The objects of a set are usually called "elements". Membership of a set is indicated by the notation $\in$. We note that this notation is a variant of the fifth letter of Greek Alphabets "ε" called epsilon. Thus if we want to say "2 is an element of the set $\mathbb{Z}$", we write $2 \in \mathbb{Z}$ and we read it as 2 belongs to $\mathbb{Z}$. The notation **"$\notin$"** stands for "is not an element of". Thus $\sqrt{3} \notin \mathbb{Z}$ is read also as "$\sqrt{3}$ is not an element $\mathbb{Z}$" or "$\sqrt{3}$ does not belong to $\mathbb{Z}$".

Description of sets:

Sets can be described in two different ways:

1. Enumeration: Here we simply list the elements of the set. When the set is infinite and there is no point of any confusion we use the three dots "…" to convey the information for the remaining elements of the set.

Example: S = {-4, 0, 4, 3}

$\mathbb{Z}$ = {…, -4, -3, -2, -1, 0, 1, 2, 3, 4,…}

2. Description:

Example: S = $\{x \in \mathbb{Z} \mid (x+4)x(x-4)(x-3) = 0\}$,

where $\mathbb{Z}$ is the set of integers.

Note that we have discussed earlier different types of numbers. The notations for these sets are:

The Set of Positive Integers: = $\mathbb{N}$

The Set of all Integers: = $\mathbb{Z}$

The set of Rational Numbers: = $\mathbb{Q}$

The set of Real Numbers: = $\mathbb{R}$

The set of Complex Numbers: = $\mathbb{C}$

Empty set and Universal set: The set which contains no element is called an empty set and is denoted by the notation $\emptyset$ called phi. The idea of a universal set is very important. Whenever one is talking of anything one has to keep in mind what is the universal set under consideration. Bertrand Russell was the one who emphasized the need of universal set. He also demonstrated that if you don't specify the universal set of your situation you may create paradoxical statements.

Universal set is generally denoted by letter U. Some time context makes it clear as to what is your universal set under consideration.

<u>Operations on sets:</u>

1. <u>Union</u>: The union of two sets A and B is denoted by $A \cup B$ and is the new set consisting of all the elements which are either in A or B. Thus,

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

2. <u>Intersection</u>: The intersection of two sets A and B is denoted by $A \cap B$ and is the new set consisting of elements which are both in A as well as B. Thus,

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

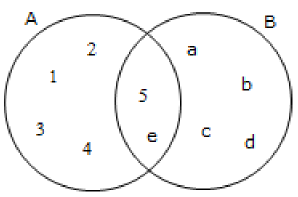
3. $A - B$ or $A \setminus B$: The set $A - B$ is the set of all elements of A which are not in B. In other words

$$A - B = A \setminus B = \{x \mid x \in A \text{ but } x \notin B\}$$

4. <u>Complement</u>: The complement of set A is the set of all elements of the universal set U which are not in A. Complement of a set A is denoted by A' or $U \setminus A$ or $\tilde{A}$. Note that that complement of U is $\emptyset$. That is $U' = \emptyset$. And $\emptyset' = U$.

<u>Venn Diagrams:</u> Sets are represented by Venn Diagrams.

<u>Venn Diagram for Union</u>:

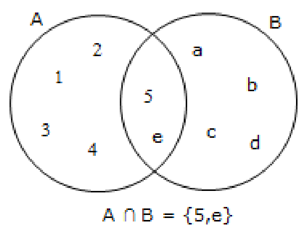


A ∪ B = {1,2,3,4,5,a,b,c,d,e}

Here, $A = \{1, 2, 3, 4, 5, e\}$, $B = \{a, b, c, d, e, 5\}$. And therefore

$A \cup B = \{1, 2, 3, 4, 5, a, b, c, d, e\}$

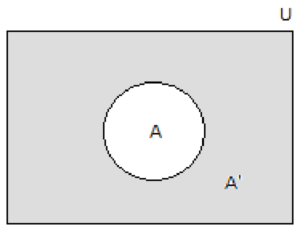
Venn Diagram for Intersection:



Here, $A = \{1,2,3,4,5,e\}$, $B = \{a, b, c, d, e, 5\}$. And therefore

$A \cap B = \{5, e\}$

Venn Diagram for Complement:



Here the universal set U is represented by points inside the rectangle. And the set A here is represented by points inside the circle which is inside the rectangle. Here A' the complement of A is denoted by the shaded part.

Subset and Equal Sets:

Subset: We say that set A is a subset of set B, if each element of A is also an element of B. This is denoted by writing $A \subseteq B$.

Equal Sets: Two sets are equal if each element of set A is in set B and also the other way (each element of set B is in set A). Thus

$$A = B \textit{ if and only if } A \subseteq B \textit{ and } B \subseteq A \text{ .}$$

Clearly set {a, b} is equal to set {b, a}, as each is a subset of the other. This means that, the order in which the elements are written in a set is immaterial.

Some of the important results related with these notions are:

1. For any set A, $\emptyset \subseteq A$
2. For any set A, $A \cap \emptyset = \emptyset$
3. For any set A, $A \cup \emptyset = A$
4. DeMorgan's Laws: $(i)\ (A \cap B)' = A' \cup B'$ $(ii)\ (A \cup B)' = A' \cap B'$

4.Summary

Familiarity with the number system was very briefly developed. We, of course, concentrated more on real numbers which are most useful for understanding the mathematics of Economics.

Logic is implicitly hidden in any statement and poof that one talks of in Mathematics. We therefore gave very brief and quick bird's eye view of logic along with the concept and methods of proof which are generally an integral part of any mathematical discussion.

The phrase "Equations" and the notation "=" were carefully discussed.

The rudiments of Set theory were discussed up to the point from where satisfactory understanding of relations and functions can be built up.