B. ARCHITECTURE

MECHANICS OF STRUCTURE – II (AR6301)

DEFLECTION OF BEAMS – CANTILEVER BEAM

Lecture - 9

Double Integration Method:

Double integration method will be integrating twice the bending equation and then will be arriving at deflection of required values or points. Also we occur the slope at the required points.

In the moment area method we will be drawing the bending moment diagram with the help of bending moment diagram of a beam we will be calculating the area of the bending moment diagram and we will also calculate the moment produced by the area and we will use that moment of area for calculating slope as well the deflection. The macaulay's method is refined method of integration which will be used exclusively or which will be most advantages in the case of simply supported beams. Where in we involve several loads and several portions has to be analyzed for deflection.

So as we discussed take a beam either it is a cantilever beam or a simply supported beam. Let us take a cantilever beam and it is subjected to the load like this and we have already studied about the supports. This is the fixed support, at fixed support there won't be any slope or deflection and the beam will get deflected like this. We call this as the deflection curve and this is one boundary point and this is another one boundary point. So at the boundary point normally we are interested in finding the deflection as well the slope. If you draw a tangent to the deflection curve and measure the angle it will represent the slope and the horizontal line represent the deflection.

In this chapter we are clearly see how to calculate the slope and deflection at different point using varies methods. So let us take the first example as,

1. Cantilever beam with point load at free end:

We are going to analyze the cantilever beam which is subjected to the point load W at the free end and let the span be l. So all this double integration method that involves bending equation or the differential equation for all flexural problems we have is,

$$EI.\frac{d^2y}{dx^2} = -Mxx$$

This is the second order equation for beam bending and when we are able to solve this differential equation finally we will get the value y which is the deflection as well we also get dy/dl which is nothing but the slope. So all we need to do is we need to substitute this expression and get the unknown deflection and slope. Again for deflection analysis the concept of bending moment become essential without the value of bending moment we can proceed with calculating the slope and deflection. The equation clearly shows that Mxx is the bending moment at any section xx. So let us take this example of cantilever beam and consider a section xx at a distance x from the free end.

We all know that bending moment at xx is equal to normally we proceed from the right hand for a cantilever beam. So similarly let us proceed from the right hand and the bending moment will be equal to load W and the distance x and we need to necessarily use standard sign convention. Here it is right clockwise is negative therefore it will give a negative value.

$$Mxx = -Wx$$

When we substitute the value of Mxx in the general equation we get,

$$EI.\frac{d^2y}{dx^2} = -(-Wx)$$

$$= Wx ----------(1)$$

In the double integration method we will be integrating equation (1) two times to get the unknown slope as well the deflection. So when we integrate equation one with respect to x we get,

$$EI.\frac{dy}{dx} = \frac{Wx^2}{2} + c_1 ------(2)$$

Where C_1 is the constant of integration and then we need to integrate again equation (2) with respect to x, we get

$$EI.\frac{Wx^3}{6} + c_1x + c_2 --------(3)$$

So we involve two constants of integration and the constants of integration shall be obtained using the well known boundary conditions of the problem.

Take a cantilever beam subjected to point load. As we discussed initially it will deflect like this. The beam will be having zero deflection at the fixed end and it will be having maximum at the load point at the free end as shown in the figure and we are measuring distance from the right end. So we can generate the boundary condition as,

When $x = l, \frac{dy}{dx} = 0 --------(i)$

When $x = l, y = 0 -------(ii)$

This indicates that there is no slope and deflection at A which is fixed. Now this two boundary conditions have to be substituted in the appropriate equation which we developed above to get the values of concepts of integration. So let us substitute boundary condition (1) in equation (2) we get,

$$0 = c_1 + \frac{wl^2}{2} \Rightarrow c_1 = -\frac{wl^2}{2}$$

Using boundary condition (2) in equation (3), we get

$$0 = c_2 + \frac{wl^3}{6} - \frac{wl^2}{2}x$$

$$\Rightarrow c_2 = \frac{wl^3}{2} - \frac{wl^3}{6} = \frac{wl^3}{3}$$

Hence we get the general equation for slope as,

$$EI.\frac{dy}{dx} = \frac{wx^2}{2} - \frac{wl^2}{2} --------(A)$$

The general equation for deflection is

$$EI.y = \frac{wx^3}{6} - \frac{wl^2}{2}x + \frac{wl^3}{3} --------(B)$$

The reason why we call this as general equation is we can obtain the slope value as well the deflection value at any point of the beam in between the A and B using this general equation. For example If I am interested in finding the slope at free end I need to put x = 0 in the equation (A) i.e., the slope equation. If I want the deflection at the free end I need to substitute x = 0 in the deflection equation that is equation (B). Similarly if I want deflection and slope at any intermediate point say for example I want slope at the midpoint of the beam. I need to substitute x = l/2 in the slope and deflection equations to get the slope and deflection values at the mid-points of the beam respectively.

Slope & Deflection at the Free End:

Now let us find the slope and deflection at the free end. So to get the slope at the free end B, put x = 0 in the equation (A),

$$\therefore EI.\frac{dy}{dx} = -\frac{Wl^2}{2}$$

$$\therefore \frac{dy}{dx} = -\frac{Wl^2}{2EI}$$

$$\Rightarrow \theta_B = -\frac{Wl^2}{2EI}$$

Next to find the deflection at the free end we need to put x = 0 in equation (B),

$$\Rightarrow E.I.y = \frac{Wl^3}{3}$$

$$\therefore y = \frac{Wl^3}{3EI}$$

$$y_B = \frac{Wl^3}{3EI}$$

Where EI represents the flexural rigidity of the beam and the similar term we have AE is also a flexural rigidity. But that is applicable in actually loaded purpose. In case of flexural load EI is the flexural rigidity.

The next case is the cantilever beam with the uniformly distributed load. So let us take a cantilever beam with uniformly distributed load over the entire span. The beam bending equation will be,

$$EI.\frac{d^2y}{dx^2} = -Mxx$$

We need to obtain the bending moment at a section xx, so let us consider a section xx at x distance from the free end. So if you take bending moment at xx to the right side of the section we have load of $W \times x$ and the load acts at a distance of $x/2$. Therefore the bending moment will be

$$Mxx = -(wx) \times \frac{x}{2}$$

$$= -\frac{wx^2}{2}$$

$$EI.\frac{d^2y}{dx^2} = \left(\frac{wx^2}{2}\right) --------(1)$$

Here the negative sign is because the moment has right clockwise and as for our sign convention right clockwise is negative. We need to integrate this equation two times to get the general equation for slope and the deflection. So integrating equation (1) with respect to x, we get

$$EI.\frac{dy}{dx} = \frac{wx^3}{6} + c_1 ------(2)$$

Integrating (2) with respect to x, we get

$$EI.y = \frac{wx^4}{24} + c_1x + c_2 ----(3)$$

So here also we have involves two constants of integration and boundary conditions need to be substituted to get the values of concepts of the integration. So the boundary conditions will be

When $x = l, \frac{dy}{dx} = 0 - - - - - - (i)$

When $x = l, y = 0 - - - - - (ii)$

So substituting the boundary condition we will be able to get the constants of integration. So substituting boundary condition (1) in equation (2) we get,

$$0 = c_2 + \frac{wl^3}{6} \Rightarrow c_1 = -\frac{wl^3}{6}$$

Using boundary condition (2) in equation (3) we get,

$$0 = c_2 + \frac{wl^4}{24} - \frac{wl^3}{6} l$$

$$\Rightarrow c_2 = \frac{wl^4}{6} - \frac{wl^4}{24} = \frac{4wl^4 - wl^4}{24}$$

$$= \frac{3wl^4}{24} = \frac{wl^4}{8}$$

Hence the general equation for slope is

$$EI.\frac{dy}{dx} = \frac{wx^3}{6} - \frac{wl^3}{6} - - - - - - - - (A)$$

The general equation for deflection is

$$EI.y = \frac{wx^4}{24} - \frac{wl^3}{6} x + \frac{wl^4}{8} - - - - - - (B)$$

Once we have the general equation for the slope and deflection and then we can find slope and deflection at any intermediate point or at any required point. So to get slope at the free end i.e., at B put x = 0 in equation (A).

$$\therefore EI.\frac{dy}{dx} = -\frac{wl^3}{6}$$

$$\therefore \frac{dy}{dx} = -\frac{wl^3}{6EI}$$

$$\Rightarrow \theta_B = -\frac{wl^3}{6EI}$$

To get deflection at B put x = 0 at deflection equation thereby we get the deflection at the free end,

$$\Rightarrow E.I.y = \frac{wl^4}{8}$$

$$\therefore y = \frac{wl^4}{8EI}$$

$$y_B = \frac{wl^4}{8EI}$$

Now when we have a cantilever beam subjected to uniformly distributed load over the entire span the deflection will be like this. So slope at be will be equal to $\theta_B = -\frac{wl^3}{6EI}$ and deflection at will be equal to $y_B = \frac{wl^4}{8EI}$. So this are the standard loading case for which we have obtained in the slope and deflection with the use of double integration method. The method double integration method means from the names itself that we need to do two times in the integrations were in we involve two constant of integration which are solved using the boundary conditions of the problem. In the case of cantilever beam, we will have the slope and deflection at the fixed using boundary conditions constant of integration can be found substituted to get the general equations for slope and deflection, which get the values for the two standard cases like this.

Example Problem for Slope & Deflection:

Now we will move on to a numerical example say we have given a cantilever beam of 120mm wide and 150mm deep is 1.8m long. Determine the slope and deflection at the free end of the beam, when it carries a point load at 20kN at its free end. Take E = 200 Gpa and slope at free end B.

Solution:

So we are given with a cantilever beam of 120mm wide and 150mm deep and its span is 1.8 long which is subjected to a point load of 20kN at the free end. Here the W is the point load which is equal to 20kN. And the young's modulus is equal to 200 Gpa.

$$= -\frac{wl^2}{2EI}$$

$$W = 20kN = 20 \times 10^3 N$$

$$= 2 \times 10^3 N / mm^2$$

Substituting all this values we can get the moment of inertia I value as,

$$I = \frac{bd^3}{12} = \frac{120 \times 150^3}{12}$$

$$= 33.75 \times 10^6 mm^4$$

Therefore θ_B will be equal to,

$$= -\frac{wl^2}{2EI}$$

$$\therefore \theta_B = \frac{-20\times10^3\times1800^2}{2\times2\times10^5\times33.75\times10^6}$$

$\theta_B = -0.0048 radians$.

Next we need to find the deflection at free end,

$$y_B = \frac{Wl^3}{3EI}$$

Where we have the values of W, L, E, I as,

$W = 20kkN = 20\times10^3 N$

$L = 1.8m = 1800mm$

$E = 2\times10^5 N/mm^2$

$I = 33.75\times10^6 mm^4$

$$\therefore y_B = \frac{20\times10^3\times1800^3}{3\times2\times10^5\times33.75\times10^6}$$

$y_B = 5.76mm$.

Example:

A cantilever beam of 3m long carries a point load of 20kN at a distance of 2m from the fixed end. Determine the slope and deflection at the free end of the cantilever. Take $EI = 8\times10^{12} N/mm^2$

Solution:

$$\theta_c = -\frac{Wl^2}{2EI}$$

To find the slope at free end first the slope at C should be found with the given data that,

$$W = 20kN = 20 \times 10^3 N$$

$$EI = 8 \times 10^{12} N/mm^2$$

$$\therefore \theta_c = -\frac{20 \times 10^3 \times 2000^2}{2 \times 8 \times 10^{12}}$$

$$\theta_c = -0.005 radians$$

$\theta_c = \theta_B$, since there is no load beyond 20kN load. To find the deflection at free end first we need to find the deflection at C,

$$y_c = \frac{Wl^3}{3EI}$$

$$W = 20 \times 10^3 N$$

$$l = 2m = 2000mm$$

$$EI = 8 \times 10^{12} N/mm^2$$

Substituting this known values of W, l and EI we can find the deflection at c,

$$\therefore y_c = \frac{20 \times 10^3 \times 2000^3}{3 \times 8 \times 10^{12}} = 6.67mm$$

$$y_B = y_c + \theta_c \times B_c$$

$$= 6.67 + 0.005 \times 1 \times 10^3$$

$$y_B = 11.67 mm$$