B. ARCHITECTURE

MECHANICS OF STRUCTURE – II (AR6301)

COLUMNS – LONG COLUMNS

Lecture - 15

Theory of Long Columns:

We have already derived the effective length of columns or the critical load carried by the columns for three end conditions. We have four end conditions which are both ends hinged, both two end fixed, one end fixed and the other end free and then one end hinged and the other end fixed. We have arrived the critical load for three cases. Then we will see how to find the Euler's crippling load or Euler's buckling load for the other end condition which is one end fixed and other end hinged.

So one end is hinged here and the other end is fixed here. In case of hinged end we will be having horizontal reactions also in addition to the vertical reactions. So that the horizontal reaction is H as shown in the diagram and we will have the load P as applied here. So the column will be deflecting in this fashion. The deflection will be zero at the fixed end as well at the hinged end and the slope at the fixed end will be zero. But we will be having slope at hinged end and we will be calculating the bending moment at any section xx. So let the deflection at this point be y and the bending moment at xx will be equal to the H multiplied by the distance say (l-x). We have considered x from the base and the remaining distance will be (l-x) so the moment produced by H with respect to x will be $H \times (l-x)$ and the nature will be clockwise. Whereas the actual load P produces an anticlockwise moment about this point. The value equal to $-P \times y$. Therefore bending moment due to critical load P will be $-P \times y$ and moment due to horizontal crust will be $H \times (l-x)$. Therefore,

$$\therefore EI.\frac{d^2y}{dx^2} = H(l-x) - P \times y$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}.y = \frac{H(l-x)}{EI}$$

The things which we need to remember or we need to understand clearly is again considering or finding bending moment at the section or given point. Once we are conversion with finding the bending moment then we can very easily forms the differential equation. So the bending moment concepts which we studied in unit 1 and subsequently we applied the bending stress computation as well in the deflection calculation and here again we need the concept of bending moment. As such the concept of bending moment is very essential for the analysis as well in the design structures which we will be studying subsequently. So one has to be through in the concept of moment, the moment developed is $H \times (l-x)$ and $-P \times y$.

$$\therefore EI.\frac{d^2y}{dx^2} = H(l-x) - P \times y$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}.y = \frac{H(l-x)}{EI}$$

The general solution for this differential solution will be equal to

$$y = A\cos\left(x\sqrt{\frac{P}{EI}}\right) + B\sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{H(l-x)}{P}$$

Now we have several boundary conditions and to substitute the boundary conditions find the value of A and B, then we will be having the equation for critical load of the column with one end fixed and the other end hinged. So when x=0 and y=0 i.e., at fixed point we have deflection equal to zero. So substituting that we get,

$$\Rightarrow 0 = A + \frac{Hl}{P}$$

$$\therefore A = -\frac{Hl}{P}$$

Then we substitute the other condition at the fixed end the slope is also zero. That is slope in the sense dy/dx. So before substituting in the boundary condition we should have the value of dy/dx. We have the value of y so dy/dx can be evaluated by differentiating it.

$$\frac{dy}{dx} = -A\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right) + B\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right) - \frac{H}{P}$$

This will give as the value,

$$\Rightarrow 0 = B\sqrt{\frac{P}{EI}} - \frac{H}{P}$$

$$\therefore B = \frac{H}{P} \times \sqrt{\frac{EI}{P}}$$

Then we substitute the other boundary condition that the column is fixed here and it is hinged here. So at the hinged end again the deflection will be zero. So we can very well make use of that boundary condition that is when x=l and y=0.

$$\Rightarrow 0 = -\frac{Hl}{P}\cos\left(l\sqrt{\frac{P}{EI}}\right) + \frac{H}{P} \times \sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right)$$

On simplifying the above equation we will be getting,

$$\frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right) = \frac{H}{P}\cos\left(l\sqrt{\frac{P}{EI}}\right)$$

$l\sqrt{\frac{P}{EI}}$ Should be in radians

So this is possible when $\Rightarrow l\sqrt{\frac{P}{EI}} = 4.5$. That is when $\tan\theta = \theta$ the condition is possible and we have taken θ radian. So when we put the 4.5 for the above equation then we will get the concept of,

$$\frac{H}{P}\sqrt{\frac{EI}{P}}\sin\left(l\sqrt{\frac{P}{EI}}\right) = \frac{H}{P}\cos\left(l\sqrt{\frac{P}{EI}}\right)$$

$$\therefore \tan\left(l\sqrt{\frac{P}{EI}}\right) = l\sqrt{\frac{P}{EI}}$$

$$\Rightarrow l\sqrt{\frac{P}{EI}} = 4.5$$

On squaring on both side we will get the value,

$$\Rightarrow l^2 \frac{P}{EI} = 20.25$$

Hence we wish to have this crippling load in the form of

$$\therefore P = \frac{20.25EI}{l^2} = \frac{2\pi^2 EI}{l^2}$$

$$\therefore P = \frac{2\pi^2 EI}{l^2}$$

So the crippling load or the buckling load in case of column with one end fixed and other end hinged is $\frac{2\pi^2 EI}{l^2}$.

Then we can write the basic formula of Euler's crippling load as,

$$\therefore P = \frac{2\pi^2 EI}{cl^2}$$

Where 'c' depends on different end conditions so C =1 for columns with both ends hinged. C=4 for column with one end fixed and other end free. C=1/4 in the cases that both ends fixed and in one end fixed and other end hinged. So we can remember in this manner also for the crippling load for different end condition.

The another way for arriving at the crippling load easily is we can remember the effective length of the column for different end conditions. In general formula it is $\frac{\pi^2 EI}{(l_{eff})^2}$. So we can very well remember this formula and the leH takes the value of L when both the ends are hinged and leH is equal to 2L when one end fixed and other end free and when both the ends are fixed then leH takes the value of l/2 and then leH takes the value of $\frac{l}{\sqrt{2}}$ if one end fixed and other end hinged. We can remember the crippling formula in this manner also. So the general formula is,

$$\therefore P = \frac{\pi^2 EI}{(l_{eff})^2}$$

On substituting the values of l_{eff} we will get, for both ends are hinged $(l_{eff} = 1)$.

$$\therefore P = \frac{\pi^2 EI}{l^2}$$

When one end fixed and other end free then $(l_{eff} = 2l)$

$$\therefore P = \frac{\pi^2 EI}{(2l)^2} = \frac{\pi^2 EI}{4l^2}$$

When both ends are fixed $(l_{eff} = l/2)$

$$\therefore P = \frac{\pi^2 EI}{(l/2)^2} = \frac{4\pi^2 EI}{l^2}$$

When one end fixed and other end hinged $(l_{eff} = l/\sqrt{2})$

$$\therefore P = \frac{\pi^2 EI}{(l/\sqrt{2})^2} = \frac{2\pi^2 EI}{l^2}$$

So these are the things one need to remember in case of long columns. In long column the formula depends on the different end conditions.

Examples Problem for Euler's Column Theory:

So we need to derive the crippling load for different end condition. We should be in a position to derive the formula as well we should be in a position to apply the formula in numerical examples.

Example 1:

A steel rod 5m long and 40mm diameter is used a column with one end fixed and other end free. Determine the crippling load by Euler's formula. Take E=200Gpa.

Solution:

Here we have a steel rod of 5m long and 40mm diameter. It is used as a column with one end fixed and the other end free. So we need to clearly notice the end condition of the column before solving the problem. We need to determine the crippling load by Euler's formula. Here the young's modulus is given as 200Gpa.

We know that the Euler's crippling load is given by,

$$P = \frac{\pi^2 EI}{{l_{eff}}^2}$$

Let us substitute the terms individually, the young's modulus is given as 200Gpa so it is equal to,

$$E = 200Gpa$$

$= 200 \times 10^9 N / m^2$

$= 2 \times 10^5 N / mm^2$

For one end fixed and other end free the l effective will be equal to $(l_{eff} = 2l)$.

$$\therefore P = \frac{\pi^2 EI}{(2l)^2} = \frac{\pi^2 EI}{4l^2}$$

So we have E value and I value. Hence we have the length also so substituting all this in the general equation, in case of circle the length is 5m long and the diameter of the rod is 40mm. Using the moment of inertia formula we get the I value as,

$$I = \frac{\pi d^4}{64} = \frac{\pi \times 40^4}{64} = 125600 mm^4$$

$$l = 5m = 5000mm$$

So it is very easy to arrive at the Euler's crippling load if we are familiar with the different end conditions and if we are conversion in finding or computing the moment of inertia.

$$\therefore P = \frac{\pi^2 EI}{(2l)^2} = \frac{\pi^2 EI}{4l^2}$$

$$\therefore P = \frac{\pi^2 \times 2 \times 10^5 \times 125600}{4 \times (5000)^2} = 2476.7N$$

$$= 2.48kN$$

Then here we have another one problem of hollow section.

Example 2:

A hallow alloy tube 4m long with external and internal diameters of 40mm and 25mm respectively was found to extend by 4.8mm under a tensile load of 60kN. Find the buckling load for the tube with both ends pinned. Also, find the safe load on the tube taking a factor of safety of 5.

Solution:

There is a hollow column with the external diameter 40mm and the internal diameter is 25mm and it found to extend by 4.8mm under a tensile load of 60kN. We need to find the buckling load for the tube with both ends pinned. Here pinned means hinged and we are also asked to find the safe load on the tube taking a factor of safety of 5.

The general formula is $= \frac{\pi^2 EI}{l_{eff}^{\ 2}}$. Here L effective can be calculated based on the given end conditions. Here both ends are hinged so in this case the effective length is equal to given length itself.

$$l_{eff} = l = 4mm = 4000m$$

Then we need to calculate the value of I and young's modulus. Sometimes young's modulus will be given directly or will be given as extend value like this. Here we found the extend value of 4.8mm under the tensile load of 60kN. We very well know how to calculate young's modulus when we have the load and the displacement or the extension. The young's modulus will be,

$$E = \frac{Pl}{Adl}$$

$$= \frac{60 \times 10^3 \times 4000}{765.375 \times 4.8}$$

$$= 65327.45 N / mm^2$$

The area for the hollow circular section it will be given by,

$$= \frac{\pi(D^2 - d^2)}{4}$$

δl will be the extension and it is given as 4.8mm thereby we can calculate the value of E. So young's modulus is know and the 'I' value can be computed. Then L effective is equal to the given length itself in case of both the ends are hinged. On substituting this we will get the values,

$$\therefore P = \frac{\pi^2 \times 65327.45 \times 1.064 \times 10^5}{4000^2}$$

$$= 4283.3 N = 4.28 kN$$

In this problem we are asked to calculate the safe load. So sometimes we will be having direct load or working load or the safe load. The safe load will be given by the crippling load which we calculated using the Euler's crippling load formula divided by the factor of safety. So factor of safety is given in the problem as 5.

$$safe.load = \frac{Crippling.load}{Factor.of.safety}$$

$$= \frac{4.28}{5}$$

$$= 0.856kN$$

We need a safe load the column can carry or the rod can carried with the column.

Example 3:

Compare the ratio of the strength of a solid steel column to that of hollow column of the same cross sectional area. The internal diameter of the hollow column is ¾ of the external diameter. Both the column have same length and are pinned at both ends.

Load carrying capacity of solid column $P_s = \frac{\pi^2 E I_s}{{l_{eff}}^2}$

Solution:

This will be the problem of calculating the ratio of strength of the solid steel column so that of the hollow column of a same cross section area. This will be the problem of interest that we will be in a position to find whether a hollow section will be enough or a solid section is enough to carry the load. Thereby we can go for safety section of the material of the column. So we need to compare the ration of the strength of the solid steel column so that of the hollow column of a section the area.

The internal diameter of the hollow column is ¾ of the external diameter. Both the columns are same length and same end i.e., both are pinned at the both ends. So load carrying capacity of hollow column is given by,

$$P_H = \frac{\pi^2 E I_H}{l_{eff}^2}$$

Then the ratio, we are interested in finding the ratio of the solid column to the hollow section. So it will be given by,

$$\therefore \frac{P_s}{P_H} = \frac{I_s}{I_H} (E \,\&\, leH.being.the.same)$$

When we find the ratio of the moment of inertia of the solid section to hollow section that will indicate the ratio of the strength of the two columns. So it will be given as,

$$I_s = \frac{\pi D^4}{64}$$

Where D is the diameter of the solid column and in case of solid section I_H will be equal to,

$$I_H = \frac{\pi}{64}(D_0^4 - D_i^4)$$

Where D_0 stands for outer diameter of hollow column and D_i stands for inner diameter of hollow column so the inner diameter is given as three fourth the outer diameter.

$$D_i = \frac{3}{4} D_0$$

It is nothing but 0.75 of the outer diameter. So this ratio will normally be given in the problem. So what will be the internal diameter in terms of the outer diameter that will be given as,

$$\therefore \frac{I_S}{I_H} = \left(\frac{\pi D^4}{64}\right) \div \left(\frac{\pi}{64} 0.684 D_0^4\right)$$

$$= \frac{D^4}{0.684 D_0^4} \text{-------}(1)$$

It is also given that both has the same cross-sectional area. So area of solid is equal to area of hollow section.

$$A_s = \frac{\pi D^2}{4}; A_H = \frac{\pi}{4}(D_0^2 - D_i^2)$$

Where D_0 is the outer diameter and D_i is the inner diameter of the hollow section.

$$\Rightarrow \frac{\pi D^2}{4} = \frac{\pi}{4}(D_0^2 - (0.75 D_0)^2)$$

$$\Rightarrow D^2 = 0.4375 D_0^2$$

$$\therefore \frac{D^2}{D_0^2} = 0.4375$$

On squaring both side we get,

$$\frac{D^4}{D_0^4} = 0.191$$

So substituting this in equation (1) we get,

$$\frac{P_S}{P_H} = \frac{I_S}{I_H} = \frac{D^2}{0.684D_0^4}$$

$$= \frac{0.191}{0.684} = 0.28$$

$$\frac{P_S}{P_H} = \frac{1}{0.28} = 3.57$$

$$\therefore P_H = 3.57P_s$$

This is the thing which we have arrived at the solid column and the hollow columns. So this type of comparison will also be given and sometimes we will be asked to find the percentage saving in material by using the hollow column. Because from this problem we understand that the hollow column is able to take 3.57 times the strength of the solid column. So in that case the hollow column will be economical thereby we go for saving the material and we also given sections such as compound sections like this. Say you are given a rectangular, circular or hollow circular section. Similarly we will also be given composite sections like this.

Example 4:

An 'I' section 400mm x 200mm x 20mm and 6m long is used as a strut with both ends fixed. Determine Euler's crippling load. Take E=200 Gpa.

Solution:

Here the load calculation by Euler's formula is given by,

$$P = \frac{\pi^2 EI}{l_{eff}^2}$$

We will calculate each and every term. So young's modulus is give as 200 Gpa.

$$E = 200 Gpa$$

$$= 200 \times 10^9 N/m^2$$

$$= 2 \times 10^5 N/mm^2$$

We need to calculate the moment of inertia should be calculated for both $I_{xx} \& I_{yy}$. Because we have studied the both the case that is bending moment about x axis and y axis and then we need to decide which should be used for the load calculations. If you take I_{yy} it can be computed easily because the axis yy passes through the c.g of all the elements. The 'I' section comprises top flange, bottom flange and the width. Hence self moment of yy axis will be useful in computing the moment of inertia with respect to yy. So it is given by,

$$I_{yy} = \frac{d_1 b_1^3}{12} + \frac{d_2 b_2^3}{12} + \frac{d_3 b_3^3}{12}$$

$$= \frac{20 \times 200^3}{12} + \frac{360 \times 20^3}{12} + \frac{20 \times 200^3}{12}$$

$$= 0.269 \times 10^8 mm^4$$

In case of I_{xx} we have already constructed the bending moment for the xx section,

$$I_{xx} = \frac{BD^3 - bd^3}{12}$$

Where we have B is equal to 200mm, D is equal to 400mm and the values of b and d will be,

$$b = 200 - 20 = 180mm$$

$$d = 400 - 2 \times 20 = 360mm$$

$$= \frac{200 \times 400^3 - 180 \times 360^3}{12}$$

$$= 3.668 \times 10^8 mm^7$$

Once the I_{xx} will be computed using this relation. Then we need to calculate $I_{\min}$ so in this case the minimum of moment of inertia is,

$$I = I_{\min} = 0.269 \times 10^8 mm^4$$

We have both ends fixed hence the effective length will be equal to,

$$\therefore l_{eff} = l/2 = 6/2 = 3m = 3000mm$$

So it is better to express the exponential in the same so that we can compare the values of I_{xx} & I_{yy} .

$$\therefore P = \frac{\pi^2 EI}{l_{eff}^2}$$

$$= \frac{\pi^2 \times 2 \times 10^5 \times 0.269 \times 10^8}{3000^2}$$

$P = 5.89 \times 10^6 N$

$= 5890 kN$

Rankine’s Formula:

We need to study about rankien’s formula for calculating the crippling load.

Rankien’s Formula:

$$P = \frac{\sigma_C A}{1 + a\left(\frac{l_{eff}}{k}\right)^2}$$

σ_C is crushing stress of the column material

A is the cross sectional area of the column

a is the Rankien’s constant $\left(\frac{\sigma_C}{\pi^2 E}\right)^2$

l_{eff} is effective or equivalent length of the column

In case of Euler’s formula we should remember the value of young’s modulus. Sometimes young’s modulus will be given or sometimes it need to be calculated from given tension and then comes moment of inertia computation. So in moment of inertia computation we need to be familiar with moment of inertia of circular section, hollow circular section, “I” section etc and sometimes we should calculate the least moment of inertia and then L effective depends on the end condition so we should notice the end conditions of the problem. When both the ends are hinged then L effective is equal to L itself, when both the ends are fixed then L effective is equal to L/2. When one end is fixed and the other end is free then the L effective is equal to 2L and when we have one end fixed and the other end hinged then the L effective is equal to $L/\sqrt{2}$. So once we know all the values we need to substitute in the Euler's crippling load formula and then get the crippling or buckling load of the section.