B. ARCHITECTURE

MECHANICS OF STRUCTURE – II (AR6301)

COLUMNS – LONG COLUMNS

Lecture - 14

Example Problem for Stress at Corner of Pier:

We have been seeing a biaxial loaded case a masonry pier supporting a vertical load of 80kN and acts at the eccentricity of 0.5m with respect to x axis and 1m with respect to yy axis. So we need to find the stress in the corners. So stress in corner A given by

$$\sigma_A = \frac{P}{A} + \frac{M_{xx}}{Z_{xx}} - \frac{M_{yy}}{Z_{yy}}$$

So the reason for plus in the $\frac{M_{xx}}{Z_{xx}}$ is our corner A lies in the same side with respect to x axis as well the load is applied on the same side so it is plus and with respect to yy axis the corner A is to the left side of the axis whereas the load acts to the right side of the yy axis thereby it causes tensile stress and hence it is $-\frac{M_{yy}}{Z_{yy}}$.

When corner B is considered here we have with respect to both xx and yy axis. We have the corner as well the load being in a same side which produces compressive stress from the corner B. So plus sign for both the $\frac{M_{xx}}{Z_{xx}}$ and $\frac{M_{yy}}{Z_{yy}}$.

$$\sigma_B = \frac{P}{A} + \frac{M_{xx}}{Z_{xx}} + \frac{M_{yy}}{Z_{yy}}$$

In case the corner C is considered with respect to xx axis the corner C is on one side and the load is applied on the other side. So minus sign for $\frac{M_{xx}}{Z_{xx}}$ and with respect to yy axis the corner C as well the load is applied on the same side of yy axis and hence plus $\frac{M_{yy}}{Z_{yy}}$.

$$\sigma_C = \frac{P}{A} - \frac{M_{xx}}{Z_{xx}} + \frac{M_{yy}}{Z_{yy}}$$

Then for corner D it is with respect to xx axis the load is on one side and the corner is on the other side so minus. Then with respect to yy axis also the load is applied to the right side and the corner D lies to the left side and hence it is also minus.

$$\sigma_D = \frac{P}{A} - \frac{M_{xx}}{Z_{xx}} - \frac{M_{yy}}{Z_{yy}}$$

So with these values we will have to calculate the stress at the corners. So the stress at the corners can be computed using this, so the P is the applied load which is 80kN and A is the area, bending moment value about xx axis. Hence the values are given as,

$$P = 80kN = 80\times10^3 N$$

$$M_{xx} = P\times e_x = 80\times10^3\times0.5\times1000 = 40\times10^6 Nmm$$

$$Z_{xx} = \frac{I_{xx}}{y}, I_{xx} = \frac{bd^3}{12} = \frac{4\times3^3}{12} = 9m^4$$

$$y = \frac{d}{2} = 1.5m$$

$$Z_{xx} = 6m^3 = 6\times10^9 mm^3$$

$$M_{yy} = P\times e_y = 80\times10^3\times1\times1000 = 80\times10^6 Nmm$$

$$Z_{xx} = \frac{I_{xx}}{y}, I_{xx} = \frac{bd^3}{12} = \frac{3\times4^3}{12} = 16m^4$$

$$x = \frac{b}{2} = 2m, Z_{yy} = 8m^3 = 8\times10^9 mm^3$$

$$\sigma_A = 0.00334N/mm^2, \sigma_B = 0.23N/mm^2, \sigma_C = 0.01N/mm^2,$$
$$\sigma_D = -0.01N/mm^2, indicates(tensile)$$

Hence the stress in the corners $\sigma_A, \sigma_B, \sigma_C, \sigma_D$ can be computed using the formula which we have arrived. Then we will have to study the core or tern of a section. This is more important for identifying for low tension case. So for low tension to develop what will be the limit of eccentricity that will called as the core or tern of a section. So the eccentricity should be limited to less than or equal to Z/A.

$$e \le \frac{Z}{A}$$

So we got this condition based on the bending stress equation $\frac{P}{A}+\frac{M}{Z}$.

When the direct stress is more than the bending stress then we will involve only compressive stress and the net resultant stress. Whereas if the direct stress is less than bending stress in that case we will be involving tensile stress so for low tension to develop your direct stress should be greater or at least should be equal to M/Z and using that condition we will get eccentricity should be less than or equal to Z/A and Z is the section modulus, if you take an rectangular section Z is equal to I/Y where the I is equal to $I=\frac{bd^3}{12}\ \&\ y=\frac{d}{2}$.

Substituting these values we get,

$$Z=\frac{bd^2}{6}$$

$$A=bd$$

Hence eccentricity $e \le \frac{d}{6}$

Similarly with respect to y axis the eccentricity will be as $e \le \frac{b}{6}$.

This is the diagram of core or tern how to plot. So we have y axis and x axis. Let us keep one sixth of the width to the right side and to the left side as b/6 and keep d/6 on the y axis. For low tensile to develop the tensile strength should be equal to d/6. So if you connect this you will get what is core or the tern of a section.

Therefore the meaning of the core or a tern is if the load is concentrated or if the load is applied within the core or tern then there won't be any tensile strength in the section. So we also called this as middle third rule. Middle third in the sense say this is b/6 and b/6 therefore the width of the core will be equal to b/3. Similarly the depth of the core will be equal to d/3. So we divide into one third, the load applied within the middle third then there won't be any tensile. So this middle third will be helpful in the analysis of the dam section.

In a dam or a section say if you have rectangular or trapezoidal cross section and we will be having the weight of the dam W, the water pressure P and the upward pressure and we will having the resultant lying here. The resultant should be within the middle third of the base then there won't be any tension developed in the dam or a section. So this middle third rule for a core or a tern of a section will be helpful in that case and also in case of columns. If you don't want to develop any tension in the column then the load has to be made to act within the core or the tern of the section.

Similarly that can be proved in a circle that the similar formula should be used as $e \le \frac{Z}{A}$. Where section modulus in case of circular will be $Z = \frac{\pi d^3}{32}$ because we have the moment of inertia of the column as $I = \frac{\pi d^4}{64}$ and when it is divided by Y which is diameter divided by 2. We will be getting $Z = \frac{\pi d^3}{32}$. So that should be less than Z/A where the area and hence we will get

$$e \leq \frac{Z}{8}$$

So if you have d/8 on either side and if you connect by means of a circle then that will be a circular core or tern and that dimension will be d/4 being d/8 on either side if we add we get d/4, if core or tern of a circular sections will be d/4.

Analysis of Long Columns:

Analysis of long columns, So far we have studied short columns where the short columns are subjected to direct stress as well bending stress and now we will see the theory of long columns.

A column is set to be a long column if the slenderness ratio is more than 12. Now we should know what is slenderness means, so slenderness ratio is the ratio of unsupported length or effective length of the column to the least lateral dimension. Say we are having the columns, in the column we will be having different end condition that has been discussing subsequently. Here the unsupported length or we call as effective length to least lateral dimension. Why we use the least lateral dimension is in case of column we can also have rectangular sections, circular sections and hollow sections etc. So whatever may be the least lateral dimensions say the width is b and depth is d. The least lateral dimension is d. So the ratio of unsupported length to the least lateral dimension that will called the slenderness. So if that is less than 12 the column is set to be short column and if it is more than 12 it is called the long column or a slender column.

The failure pattern in terms of short column and the long column is the short column will be subjected to crushing stress whereas the long column will be subjected to buckling and the short column fail by crushing. So the load corresponding to the crushing stress in case of short column is crushing load and the compression member does not fail entirely by crushing but also by bending that is buckling. This happens in case of long columns. Short columns fail by crushing while long columns fail by buckling.

Euler's Columns Theory:

So Euler's has developed a theory and equation to find the buckling load of columns. That is in case of long column what will be the tripling load or buckling load of the column. So Euler's derived the bucking load of long columns based on bending stress. While deriving this equation he had neglected the effect of direct stress. So as the direct stress be the predominant in case of short columns we have neglected this and focus on bending stress and assumptions in the Euler's column are we assume that initially the column is perfectly straight and the load applied is truly axial. Then the cross section of the column is uniform throughout its length. Also the column material is perfectly elastic, homogeneous and isotropic and thus obeys Hooke's law.

While deriving the Euler's column formula he has made the following assumptions also the length of the column is very large as compared to its cross sectional dimensions. This relates to the slenderness ratio only when the slenderness ratio is greater than 12 we call it as long columns. The length of the column is very large when compared to the cross sectional dimensions. Then the shortening of column due to direct compression is neglected and the failure of column occurs due to buckling alone.

Now we will see the different end conditions of the columns. So we talk about the unsupported length or the equivalent or the effective length of the column. So the effective length of column depends on the end condition of the column. When we have both ends hinged what will be the unsupported length or the effective length. Similarly when we have both ends fixed then what will be the unsupported length and there may be cases that one end is fixed and the other end is hinged and also one end is fixed and other end free. So this are the varies end conditions of long columns and Euler have derived the bucking load for the different end conditions of the columns. We will see one by one.

Euler’s Crippling load or buckling load:

Euler’s crippling load or bucking load in case of both ends hinged. Say we have column both ends hinged and the load is applied so the column will buckle like this and will be subjected to bending like this and if I take any section section x from here then the deflected shape will be y. So this is the column condition in both ends of the column are hinged. We will arrive finally at what will be the crippling load or buckling load based on Euler’s column formula and for this we will use bending equations which we have used for arriving at the deflection of hinge that we very we remember that $EI\frac{d^2y}{dx^2} = M_{xx}$. So we need to obtain the bending moment at xx and it will be equal to the moment produced by the load P multiplied by the distance y and we have put minus sign because it produces anticlockwise moment.

$$EI\frac{d^2y}{dx^2} = -P \times y$$

$$\Rightarrow EI\frac{d^2y}{dx^2} + P \times y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = 0$$

This is a second order differential equation involving y that is deflection or the column. So the general solution to this second order differential equation will be in the form of particular integral and complementary function. So if you substitute the particular integral and the complementary function the solution to the above differential equation will be,

$$y = A\cos\left(x\sqrt{\frac{P}{EI}}\right) + B\sin\left(x\sqrt{\frac{P}{EI}}\right)$$

Where A and B are constants of integration which has to be found using the boundary conditions x = 0 and y = 0.

The column bucks like this and we take x from the bottom so at x=0 and y=0 there is a boundary condition. So when you substitute this boundary condition we will get A=0 and another boundary condition is at x=l and y=0. At the other end that is at the hinged end also the deflection is zero therefore at x=l and y=0. Using this boundary condition we get,

$$\Rightarrow 0 = B\sin\left(l\sqrt{\frac{P}{EI}}\right)$$

If this is the case then either B should be 0 or $\sin\left(l\sqrt{\frac{P}{EI}}\right)$ should be equal to zero and B can't be able to zero and let,

$$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0$$

So this can be possible when $l\sqrt{\frac{P}{EI}}$ takes the value of

$$\Rightarrow \left(l\sqrt{\frac{P}{EI}}\right) = \pi, 2\pi, 3\pi, etc$$

Taking the least significant value that is π for $l\sqrt{\frac{P}{EI}}$ and squaring on both sides we get,

$$l^2\frac{P}{EI} = \pi^2$$

$$\therefore P = \frac{\pi^2 EI}{l^2}$$

So the crippling load formula for columns with both ends hinged will be,

$$P_{C_n} = \frac{\pi^2 EI}{l^2}$$

If one is interested in calculating the crippling load or buckling load we need to have young's modulus of the material of the column then moment of inertial of inertia of the column and then the effective length l or unsupported length. So once these are known we can calculate the crippling load. Similarly the Euler's crippling load or buckling load in case of column with one end fixed and the other end free. So imagine we have this P as deflection and the distance be a. And at this point the deflection will be y. So in this case P is the critical load, y is the deflection at any section x. Then bending moment at this point will be equal to,

$$EI\frac{d^2y}{dx^2} = Pa - P \times y$$

$$\Rightarrow EI\frac{d^2y}{dx^2} + P \times y = Pa$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{P}{EI} \times y = \frac{P}{EI}a$$

So the general solution for the differential equation will be,

$$y = A\cos\left(x\sqrt{\frac{P}{EI}}\right) + B\sin\left(x\sqrt{\frac{P}{EI}}\right) + a$$

Here the addition term we involve is 'a' which is the deflection at the free end. Where A and B are constants of integration the boundary conditions are x=0 and y=0. So if we substitute x=0 and y=0 we get,

$$\Rightarrow 0 = A + a$$

$$\therefore A = -a$$

Here the other boundary condition in case of column with one end fixed and other end free will deflect like this. So here at the fixed end slope is zero. Therefore at x=0, dy/dx=0. Substituting this boundary condition we will get,

$$0 = B\sqrt{\frac{P}{EI}}$$

Here either B should be equal to zero or $\sqrt{\frac{P}{EI}}$ should be equal to zero and since P is not zero the load cannot be zero and hence B should be equal to zero. Therefore we can substitute the value of A as –a, and B=0. The general equation for deflection will be,

$$y = -a\cos\left(x\frac{P}{EI}\right) + a$$

$$= a\left[1 - \cos\left(x\sqrt{\frac{P}{EI}}\right)\right]$$

Another boundary condition we have is x=l and y=a. Here the deflection will be like this and the deflection at the free end will be maximum at will be equal to a. So when x=l and y=a substituting that we get,

$$\Rightarrow a = a\left[1 - \cos\left(l\sqrt{\frac{P}{EI}}\right)\right]$$

$$\therefore \cos\left(l\sqrt{\frac{P}{EI}}\right)=0$$

This implies that $l\sqrt{\frac{P}{EI}}$ should be equal to $\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2} etc$. So again taking the least significant value we will be getting the equivalent length for column that have one end fixed and other end free. In a similar way we can derive the equivalent length formula for other cases namely columns with both ends fixed and columns with one end fixed and other end hinged.

Now we will see columns with both end fixed here P is the critical load, and in case of fixed end we will be have moment in addition to load and the moment is M_0 and the column will deflect like this and let the deflection at any section xx is equal to Y as shown in the diagram. So bending moment at this point will be equal to,

$$M_0 - P\times y$$

Actually the moment produced will be P x y and we have already we have M_0 and then the bending moment at any section will be equal to $M_0 - P\times y$. Therefore,

$$EI.\frac{d^2y}{dx^2}=M_0-P_y$$

$$\frac{d^2y}{dx^2}+\frac{P}{EI}.y=\frac{M_0}{EI}$$

Where M_0 is the fixed moment or end moment at the fixed end and then the general solution of the differential equation is

$$y = A\cos\left(x\sqrt{\frac{P}{EI}}\right) + B\sin\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P}$$

Here also we involve two constants A and B which are the constants of integration. The boundary condition are x=0 and y=0. Because this is the fixed end where both the ends are fixed we will get,

$$\Rightarrow 0 = A + \frac{M_0}{P}$$

$$\therefore A = -\frac{M_0}{P}$$

$$\frac{dy}{dx} = -A\sqrt{\frac{P}{EI}}\sin\left(x\sqrt{\frac{P}{EI}}\right) + B\sqrt{\frac{P}{EI}}\cos\left(x\sqrt{\frac{P}{EI}}\right)$$

Similarly we have other two boundary conditions also that is when x=0 the slope is also zero and when x=l then the slope is zero. So when x=0 and y=0 implies,

$$\Rightarrow 0 = A + \frac{M_0}{P}$$

$$\therefore A = -\frac{M_0}{P}$$

Then substituting the boundary condition when x=0, dy/dx=0 we get,

$$\Rightarrow 0 = B\sqrt{\frac{P}{EI}}$$

Since B should be equal to zero since P is not equal to zero and differentiating the above equation we will get,

$$\therefore y = \frac{-M_0}{P}\cos\left(x\sqrt{\frac{P}{EI}}\right) + \frac{M_0}{P}$$

$$= \frac{M_0}{P}\left(1 - \cos\left(x\sqrt{\frac{P}{EI}}\right)\right)$$

Now we will substitute the other boundary condition where x=l and y=0. So we have already made use of two boundary conditions. Substituting these boundary condition we will get,

$$\Rightarrow 0 = \frac{M_0}{P}\left(1 - \cos\left(l\sqrt{\frac{P}{EI}}\right)\right)$$

$\therefore \cos\left(\left(l\sqrt{\frac{P}{EI}}\right)\right)$ should be equal to 1.

Therefore that is possible when $l\sqrt{\frac{P}{EI}} = 0, 2\pi, 4\pi, 6\pi, etc$. Then taking the least significant value as 2π we get,

$$l\sqrt{\frac{P}{EI}} = 2\pi$$

$$\therefore l^2\left(\frac{P}{EI}\right)$$

$$\therefore P = \frac{4\pi^2 EI}{l^2}$$

This will be the effective length of the both ends fixed. Similarly we can arrive the values for columns with one end free and other end hinged.