B. ARCHITECTURE

MECHANICS OF STRUCTURE – II (AR6301)

DEFLECTION OF BEAMS – SIMPLY SUPPORTED BEAM

Lecture - 12

Deflection of Uniformly Distributed Loads:

Uniformly distributed loads, we will use the macaulay's method which we learned and then we also use double integration method in case of simply supported beams with uniformly distributed loads. In continuation to the earlier example, say we have been discussing the simply supported beam with non central load. We have 1m and 1.8m which the total span is 2.8m and we have developed the equation of the slope and deflection and we need to obtain the constants of integration using the boundary condition. So the boundary conditions will be,

$$x = 0, y = 0 ------(i)$$

$$x = 2.8m, y = 0 ----(ii)$$

Using the boundary condition in the deflection equation up to dotted line, there is the first boundary condition has to be used for portion AC alone then for portion CB we can use the entire equation. So substituting that we get C_2 equal to zero and then substituting the boundary condition (2) in equation (3) we get,

$$0 = \frac{-38.57 \times 2.8^3}{6} + C_1(2.8) \Big| + \frac{60(2.8-1)^3}{6}$$

$$\therefore C_1 = 29.57$$

After obtaining $C_1 \,\&\, C_2$ we can have the general equations for slope as,

$$EI.\frac{dy}{dx} = \frac{-38.57x^2}{2} + 29.57 \Big| + \frac{60(x-1)^2}{2} ----(A)$$

The general equation for deflection is

$$EI.y = \frac{-38.57x^2}{6} + 29.57 \Big| + \frac{60(x-1)^3}{6} ----(B)$$

If you are interested in finding the maximum deflection say the simply supported beam subjected to non central point load. So it will deflect in the fashion and it need not be necessarily maximum deflection occur under the loaded point. Deflection will be maximum where slope is zero,

$$\therefore -\frac{38.57x^2}{2} + 29.57 + \frac{60(x-1)^2}{2} = 0$$

$$\Rightarrow -19.285x^2 + 29.57 + 30(x-1)^2 = 0$$

$$\Rightarrow -19.285x^2 + 29.57 + 30(x^2 + 1 - 2x) = 0$$

If you assume the slope occurs anywhere in between portion then we need to equate the slope equation up to dotted line to zero. But if you assume the slope zero or the deflection is maximum in portion BC then we can use the entire slope equation and equate to zero. So here let us assume that the deflection is maximum in portion BC and we are equating the entire slope equation to zero thereby we get the value of x as,

$$\Rightarrow -19.285x^2 + 29.57 + 30x^2 + 30 - 60x = 0$$

$$\Rightarrow -10.742x^2 - 60x + 59.57 = 0$$

$$\therefore x = \frac{60 \pm \sqrt{60^2 - 4 \times 10.742 \times 59.57}}{2 \times 10.742}$$

$$= \frac{60 \pm 32.26}{21.484}$$

$$4.29m(or)1.29m$$

Here we have the span 2.8m and we have got x equal to 4.29m or 1.29m. So this is the problem we have the load here with 1m and 1.8m and it is clear that we got the value of 1.29 which will be lying somewhere here after portion. In case if we get the value lesser than one then we need to redo the calculation by equating the slope up to the portion AC and then by substituting the value of this x then the deflection equation will be getting the maximum deflection.

Hence the deflection will be minimum at x = 1.29m from A. Then

$$\therefore EI.y_{\max} = \frac{-38.57 \times 1.29^3}{6} + 29.57 \times 1.29$$

$$+ \frac{60(1.29 - 1)^3}{6} = 24.5$$

$$EI = 4 \times 10^{12} Nmm^2$$

$$= 4 \times 10^3 kNm^3$$

$$\therefore y_{\max} = \frac{24.59}{4 \times 10^3} = 6.15 \times 10^{-3} m$$

$$= 6.15mm$$

If you are interested in finding deflection under the load point then we need to put x equal to 1m in the general equation for deflection up to the dotted line and we will be getting a value which will be definitely lesser than the value of the 6.15m and you can make try and you can find the value of deflection exactly under load point, we have found the maximum deflection.

Deflection of Several Point Loads:

Next we will proceed to several point load or two point loads in this case.

Example 3:

Say we have the simply support beam of span 14m and subjected to two point loads of 12kN at 3m and 8kN at 9.5m from the left end. Where $E = 200 Gpa, I = 160 \times 10^6 mm^4$.

Solution:

We have to compute the deflection under the load points and also the maximum deflection. The young's modulus is given as 200 Gpa and the I value is 160 X 10^6 mm^4.

First let us find the reaction component $R_A \,\&\, R_B$. So here we need to make use of the static equilibrium to obtain the reactions $R_A \,\&\, R_B$. We have find,

i) To find reactions $R_A + R_B - 12 - 8 = 0$

$$\therefore R_A + R_B$$

Using $\sum m = 0$ and the sign conventions being positive taking the moments about A,

$$12 \times 3 + 8 \times 9.5 - R_B \times 14 = 0$$

$$\therefore R_B = 8kN$$

$\therefore R_A = 12kN$

After obtaining the values of reactions then we can start writing the bending moment equations in different portions. Say if you want to develop the equation for portion AC, then bending moment at xx will be equal to this left reaction we have 12kN and the bending moment will be

$M_{xx} = 12 \times x$

$= 12x$

Then we will move the section to CD, so at this point we have we have bending moment value with the sign convention of left clockwise which is positive and the applied load of 20kN which is anticlockwise of negative and hence the value will be,

$M_{xx} = 12 \times x - 12(x-3)$

$M_{xx} = 12x - 12(x-3)$

Then we can arrive at the bending moment equation at portion DB. We have this 12kN upward reaction and downward load 12kN and 8kN. So the reaction will produce a moment of $12 \times x$ and then the applied load will produce a moment of $12 \times (x-3)$ and 8kN will produce a moment of $8 \times (x-9.5)$. So the reactions produce clockwise moment while the applied load produce anticlockwise moment so plus and minus assign accordingly.

$M_{xx} = 12 \times x - 12(x-3) - 8(x-9.5)$

$M_{xx} = 12x - 12(x-3) - 8(x-9.5)$

If you notice the three equation say the first equation be $12x$ and the second equation is $M_{xx} = 12x - 12(x-3)$ and the third equation is

$M_{xx} = 12x - 12(x-3) - 8(x-9.5)$. If you compare the equation for their portions the $12x$ being common for all the three portions and $M_{xx} = 12x - 12(x-3)$ is common for portion CD and DB. So you can write the equation like this.

To form slope and deflection equation we have,

$$EI.\frac{d^2y}{dx^2} = -Mxx = -[12x - 12(x-3) - 8(x-9.50]$$

$$EI.\frac{d^2y}{dx^2} = -[12x - 12(x-3) + 8(x+9.50] - - - - - - - - (1)$$

Here the addition portion for the portion DB is -8(x-9.5). So we need to write in this pattern we have been practicing for the earlier example also which is the example of the simply supported beam with point load and also we have seen a simply supported beam with non central load. In both the case we have been practicing the method of writing the bending moment equation portion by portion by portioning technique like this as suggested by macaulay's.

So we have $EI.\frac{d^2y}{dx^2} = -12x - 12(x-3) + 8(x+9.50$, the other procedures will be routine and we need to integrate twice the equation and then we will be getting two constants of integration. Substituting the boundary conditions we will be getting the values of constants of integration and then substituting back in the equations for deflection and slope we will get general equation for the slope and the deflection. From that we can get the slope and deflection at any required point. Integrating equation (1) with respect to x we get,

$$EI.\frac{dy}{dx} = \frac{-12x^2}{2} + C_1 \vdots + \frac{12(x-3)^2}{2} \vdots + \frac{8(x-9.5)^2}{2} ---(2)$$

If we notice the portion AC and CD we will be having the term $(x-3)\,\&\,(x-9.5)$ and they should be considered as the single term and integrated we get $\frac{x^2}{2}$ we should not separate that. Then integrating equation (2) again with respect to x we get,

$$EI.y = \frac{-12x^2}{6} + C_1 x + C_2 \vdots + \frac{12(x-3)^3}{6} \vdots + \frac{8(x-9.5)^3}{6} ---(3)$$

Then the boundary conditions, say we have simply supported beams and we have two point loads say the beam will deflect in this fashion. The deflection will be zero at the left hand and at the right hand. Let us been mention or taken as the boundary conditions as,

When $x = 0, y = 0 ------(i)$

When $x = 14, y = 0 ------(ii)$

Using these boundary conditions in the equation we can get the constants of integration. So case should be taken to use the entire equation or the equation of the dotted line. The first boundary condition will be $x = 0, y = 0$ lies in the region AC and hence the first equation need to be used up to the first dotted line alone that we get $C_2 = 0$.

Then if you use $x = 14, y = 0$ the entire equation can be made use because x = 14m covers the entire length hence the entire equation can be made use. Thereby we will get the value as,

$$0 = \frac{-12\times 14^3}{6} + C_1(14) + \frac{12(14-3)^3}{6} + \frac{8(14-9.5)}{6}$$

$$0 = -5488 + 14C_1 + 2662 + 121.5$$

$$= 193.18$$

Now we will be have the general equation for slope as,

$$EI.\frac{dy}{dx} = -6x^2 + 193.18x \vdots + 6(x-3)^2 \vdots + 4(x-9.5)^2 ----(A)$$

The general equation for deflection is given as,

$$EI.y = -2x^3 + 193.18x \vdots + 2(x-3)^3 \vdots + 1.33(x-9.5)^3 ----(B)$$

So equation A and B always form or be the general equation for slope and deflection respectively. Now we are interested in obtaining the slope at supports and deflection at the load points. To obtain the slope at supports say if I want the slope at support A, I need to substitute x = 0 in the general equation for slope up to the first dotted line and then if I want slope at B then we need to substitute x = 14m in the equation (A) and the entire equation can be made use of. Similarly if I am interested in finding deflection under the load point say if I am interested in deflection under first load I need to put x equal to the value in the problem which is 3m and I will have to make the general equation for deflection up to the first dotted line and if you want deflection at D then I will have to make use of the general equation of deflection up to second dotted line. That indicates the portions or where our point of interest lies.

So to find deflection at C we have to put x = 3m in equation (B) up to the first dotted line in which we will get the value of EI in Nmm^2 and since the load applied in kN and the distances are in meters. It is customary to have the value of EI in kNm^2.

$$EI.y_c = -2\times 3^3 + 193.18\times 3$$

$$= 525.54$$

$$\therefore y_c = \frac{525.54}{EI}$$

$$E = 2\times 10^5 N/mm^2$$
$$I = 160\times 10^6 mm^4$$
$$EI = 32\times 10^{12} N-mm^2$$
$$EI = 32\times 10^3 kN-m^2$$

$$\therefore y_c = \frac{525.54}{32\times 10^3} = 16.423\times 10^3$$

$$\therefore y_c = 16.42mm$$

When this is substitute in deflection equation we will get the deflection in meters and it is customary to explain or express always deflection in mm. So we get 16.42mm as deflection at point C and then you want deflection at D. I need to put x = 9.5m in equation (B) up to the second dotted line and then we will get,

$$EI.y_D = -2\times 9.5^3 + 193.18\times 9.5 + 2(9.5-3)^3$$

$$= 669.71$$

$$\therefore y_D = 20.93\times 10^{-3} m$$

$$\therefore y_D = 20.93mm$$

To obtain the maximum deflection we need to assume where the maximum deflection lies in the case of simply supported beam in the case of non centric load. So this is portion A, B and C, D. So here the deflection will be like this and we have obtained the deflection under load point C and D and also somewhere in between there will be maximum deflection. So from the deflection diagram you can easily judge the value of deflection where it will be maximum.

$$-6x^2 + 193.18 + 6(x-3)^2 = 0$$

$$-6x^2 + 193.18 + (6(x^2 + 9 - 6x)) = 0$$

$$-6x^2 + 193.18 + (6x^2 + 54 - 36x) = 0$$

$$36x = 247.18$$

$$x = 6.87m$$

So let us assume that the maximum deflection occurs at the portion CD and hence we can equate the slope equation up to second dotted line zero. To get the point where deflection will be maximum, so equating that to the second dotted line we get x= 6.87m and we need to make a check whether these 6.87m meets up with the assumption we had made.

$$EI.y_{\max} = -2 \times 6.87^3 + 193.18 \times 6.87 + 2(6.87 - 3)^3$$

$$= 794.58$$

$$\therefore y_{\max} = 0.0248m$$

$$\therefore y_{\max} = 24.83mm$$

So we have first portion at 3m and the second portion at 6.87m and if our x value lies well within the 9.5m then our assumption is correct. That is we have assumed the maximum deflection in between C and D. So we got 6.87 substituting that we get value maximum deflection at 24.83mm. When this value compared to the deflection under the load point it will be definitely greater because we have deflection under D is 20.93 and deflection at C is 16.42 and this maximum deflection should be definitely greater than the values obtained under the load point. So 20.83 can confirm the correction of the problem.

Analysis of Simply Supported Beams:

Now we will see the analysis of a simply supported beam with uniformly distributed load.

Example:

Simply supported beam with uniformly distributed load.

Solution:

In case of uniformly distributed load say first we should find reaction as we have discussed. So it's a uniformly distributed load spread over the entire length and naturally the reaction will be equal and it will be equal to the total load divided by 2 which is $WL/2$.

Here we don't have any portions so the macaulay's method need not be required here. We have only one portion AB. So the equation whatever we develop for the section XX will be applicable for the entire length AB.

$$RA = RB = \frac{Wl}{2}$$

Then the bending moment at XX will be say we have the simply supported beams subjected to uniformly distributed load. We have section XX and this distance is X, this reaction is $WL/2$. The load is W. So bending moment at

XX will be equal to $\frac{Wl}{2} \times x$ that is the moment produced by reaction $WL/2$ about XX and the moment produced by the udl will be $W \times x$ will be the total load and the distance will be $x/2$ as shown here. Because the resultant will act at a distance of $x/2$ from the left or from the section and the bending moment value will be,

$$M_{xx} = \frac{Wlx}{2} - \frac{Wx^2}{2}$$

Then,

$$EI.\frac{d^2y}{dx^2} = -M_{xx}$$

$$= -\frac{wlx}{2} + \frac{wx^2}{2} --------- (1)$$

Integrating equation (1) with respect to x we get,

$$EI.\frac{dy}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + C_1 ---- (2)$$

Then integrating equation (2) with respect to x we get,

$$EI.y = -\frac{wlx^3}{12} + \frac{wx^4}{24} + C_1 x + C_2 ---- (3)$$

Again comes the problem of obtaining the constants of integration. So when we are interested in constants of integration we need to substitute the boundary condition. Say we have a simply supported beam and we should know the deflection pattern of the beam so the deflection will be zero at the supports. Hence the boundary conditions are, When $x = 0, y = 0 - - - - - (i)$

When $x = l, y = 0 - - - - - (ii)$

So we can use this two deflection boundary conditions in the deflection equations and then we can obtain the constants of integration. So using the boundary condition (1) in equation (3) we get C_2=0 and using boundary condition (2) in equation (3) we get,

$$0 = \frac{-wl^4}{12} + \frac{wl^4}{24} + C_1 l$$

$$C_1 l = \frac{wl^4}{12} - \frac{wl^4}{24}$$

$$= \frac{2wl^4 - wl^4}{24}$$

$$\therefore C_1 = \frac{wl^3}{24}$$

Now we get the general equation for slope as,

$$EI.\frac{dy}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + \frac{wl^3}{24} - - - - - - (A)$$

Then the general equation for slope is,

$$EI.y = -\frac{wlx^3}{14} + \frac{wx^4}{24} + \frac{wl^3}{24}x - - - - - - (B)$$

If I am interested in finding the slope at A, at point A I need to substitute x = 0 in the general equation for slope we get,

$$EI.\frac{dy}{dx} = -\frac{wl^3}{24}$$

$$\therefore \left(\frac{dy}{dx}\right)_A = \frac{wl^3}{24EI}$$

$$\therefore \theta_n = \frac{wl^3}{24EI}$$

Then if we want to find the deflection put x = l we get,

$$-\frac{wlx^2}{4} + \frac{wx^3}{6} + \frac{wl^3}{24} = 0$$

$$\Rightarrow \frac{-x^2 l}{4} + \frac{x^3}{6} + \frac{l^3}{24} = 0$$

$$\Rightarrow \frac{-x^2 l}{4} + \frac{x^3}{6} = -\frac{l^3}{24}$$

To find maximum deflection because in the simply supported beam subjected udl will be deflecting in this pattern and we have slope at A and B. So somewhere the slope will be zero. So equate the slope equation to zero and naturally in case of symmetrical loads the deflection will be maximum at the center in case of simply supported beam with udl or in case of simply supported beam with central point load the maximum deflection will be occurring at the center and that is evident from the analysis also.

Now substituting the value x = l/2 in the deflection equation we will be getting,

$$EI.y_{\max} = \frac{-wl(l/2)^3}{12} + \frac{w(l/2)^4}{24} + \frac{wl^3}{24} \times \frac{l}{2}$$

$$= \frac{-wl^4}{96} + \frac{wl^4}{384} + \frac{wl^4}{48}$$

$$= \frac{-4wl^4 + wl^4 + 8wl^4}{384}$$

$$= \frac{5}{384} Wl^4$$

$$y_{\max} = \frac{5}{384} \frac{wl^4}{EI}$$

This is the standard loading case say we have studied simply supported beam with central point load. So the slope at supports will be $\frac{Wl^2}{16EI}$ in either case and maximum deflection occurs at the center with the value being $\frac{Wl^3}{48EI}$.

Now we have derived the simply supported beam with uniformly distributed load over the entire span and the support slope will be $\frac{Wl^3}{24EI}$ and the maximum deflection will be $\frac{5}{384} \frac{wl^4}{EI}$.

The maximum deflection is while designing the beams that may be any beam we will have to have a check over the deflection. So after designing the beam we need to check the maximum deflection using this relation. We will be knowing the load and we will be having the span and we will be having the value of young's modulus where the moment of inertia can be computed using the cross section of the beam. So we will be having the value of the deflection and we have permissible limit for any deflection. In that case we need to check whether the deflection is same.