B. ARCHITECTURE

MECHANICS OF STRUCTURE – II (AR6301)

DEFLECTION OF BEAMS – SIMPLY SUPPORTED BEAM

Lecture - 11

Macaulay's Method:

We have been subjected to the example of cantilever beam subjected to two point loads. So the area of first portion, so we have divided the bending moment diagram into triangle and rectangle like this and we had the bending moment diagram like this which comprises three portions one triangle, another rectangle and one more triangle. So first area we have computed and then the next c.g distance we need to find for area 1. So c.g distance for area 1 from this end will be 2/3 of the length which is 0.67 and then area 2 we have already computed which is rectangle and c.g distance will lie exactly at the center so that will be again ½ and it will be 1.5.

$$x_1 = \frac{2}{3} \times 1 = 0.67$$

$$A_2 = 1 + \frac{1}{2} = 1.5$$

$$A_3 = 25$$

$$x_3 = 1 + \frac{2}{3} \times 1 = 1.67$$

Once we have the area and the c.g distances then it will be easier to compute the moment area of the bending moment diagram about the point B. So moment of area of the bending moment diagram divided by EI will give the deflection at point B. We can compute deflection at B by computing the moment of area of the bending moment diagram up to point B divided by EI and that will give as,

$$\therefore y_B = \frac{10 \times 0.67 \times 20 \times 1.5 + 25 \times 1.67}{EI}$$

$$= 0.002615m$$

$$y_B = 2.61mm$$

If we want the slope up to C, say we have two point loads A, B, and C. We have computed the deflection at C and B. Now if I want $y_c \,\&\, \theta_c$. Say if I want slope at C, I need to find the area of the bending moment at C.

$$\therefore \theta_c = \frac{Area.of.BMD.upto.C}{EI}$$

$$= \frac{A_2 + A_3}{EI}$$

$$= \frac{20 + 25}{EI}$$

$$= 0.0015radian$$

Similarly if you want the moment of area of bending moment diagram up to that point, now we want at C so calculate area up to C similarly moment of area up to C. The deflection at C will be moment of area up to C divided by EI.

$$y_c = \frac{20 \times 1 + 25 \times 0.67}{EI}$$

$$= 0.001225m$$

$$= 1.225m$$

This is how we should compute the slope and deflection by moment area method.

Example Problem for Macaulay's Method:

Next we will see the concept of arriving slope and deflection in case of simply supported beams for simply supported beams the easiest method is to go by macaulay's method. In this method which is an refined double integration method which is applied to simply supported beam.

Example 1:

Simply supported beam with central point load.

Solution:

We have, $EI.\frac{d^2 y}{dx^2} = -M_{xx}$

If you consider portion AC, say we have the simply supported beam subjected to central point load. Before proceeding to bending moment equation we need to have the value of the reactions and for problems like this symmetrically loaded beam the reaction will be simply symmetry so it will be equal. So if you consider section AC the portion xx and x distance from the left hand. Then bending moment at xx will be equal to,

$$M_{xx} = \frac{W}{2} \times x$$

We have the bending moment at left side and the left clockwise is being positive. Now if you move on to portion CB, consider a portion in CB and this distance will be x and the distance between this load W and the section so that distance will be x-l/2. Therefore the bending moment at xx will be,

$$M_{xx} = \frac{W}{2} \times x - W(x - l/2)$$

Now we are ready with the equation of bending moment for portion AC and for portion CB. Now we can substitute in the general bending equation we get,

$$\therefore EI.\frac{d^2y}{dx^2} = -\left[\frac{W}{2}x\Big| + W(x - l/2)\right]$$

$$= -\frac{W}{2}x\Big| + W(x - l/2) ------(1)$$

If you notice there is a common term in the portion. The common term will be $\frac{W}{2} \times x$. While writing the equation of M_{xx} in the right side of the bending equation we can use the above format. Now we are going to proceed by the method of double integration. Integrating equation (1) we get,

$$EI.\frac{dy}{dx} = -\frac{W}{2}\frac{x^2}{2} + C_1\Big| + \frac{W(x - l/2)^2}{2} ----(2)$$

Integrating equation (2) with respect to x we get,

$$EI - y = -\frac{W}{2}\frac{x^2}{6} + C_1x + C_2\Big| + \frac{W(x - l/2)^3}{6} ----(3)$$

Here $(x - l/2)$ should be treated as a single term. Now we have the equation for slope and deflection with some constants of integration. So the constants of integration can be found by substituting the boundary conditions. We very well know the boundary conditions in case of the cantilever beam,

At $x = 0, y = 0 - - - - - - (i)$

At $x = l, y = 0 - - - - - - (ii)$

This is the way that the cantilever may deflect and these are the boundary condition for the cantilever beam. Whereas in the simply supported beam will deflect in this. So it will have some slope at A and it will have some slope at B and somewhere this slope is zero and the deflection will be maximum at the center in case of the symmetrically loaded beams and it will be somewhere where slope is zero. Now the boundary condition will be at x is equal to zero and the deflection is zero that is very well seen in the way beam deflects. Similarly at x equal to l and y equal to zero.

We can use these boundary conditions in equation (3) earlier in case of cantilever beam we substituted one slope boundary condition and one deflection boundary condition. Here we are substituting two deflection boundary conditions. Using boundary condition (1) in (3) up to partition line, $0 = C_2$. Using boundary (2) in (3),

$$0 = -\frac{W}{2}\frac{l^3}{6} + C_1 l \Big| + \frac{W(l - l/2)^3}{6}$$

The general equation for slope is,

$$EI.\frac{dy}{dx} = -\frac{Wx^2}{4} + \frac{Wl^2}{16}\Bigg| + \frac{W(x - l/2)^2}{2} - - - - (A)$$

So we have the simply supported beam AB and load is applied at point C. We need to know up to dotted line when we should use the equation and when we should use the entire equation. If we require slope value at any point in portion AC, let it be A or let it be B or anywhere in between if you want the slope value then the slope equation has to be used up to the dotted line. Say for example if I want slope at A, I need to substitute x equal to zero in this equation (A) up to the dotted line and then we will get the value of slope. Similarly if I want slope at C for that also I can use x equal to l/2 because C lies at midpoint in the equation (A) up to the dotted line.

The general equation for deflection is,

$$EI.y = -\frac{Wx^3}{12} + \frac{Wl^2}{16}x \Big| + \frac{W(x - l/2)^2}{6} ------(B)$$

If you are interested in obtaining the deflection for any point in portion AC, we can use the general equation for deflection up to dotted line and if you have any point in portion CB and if you want deflection at that point we can use the entire equation removing the dotted line, substituting appropriate the value for x.

Advantages of Macaulay's Method:

Advantage of macaulay's method is say if you not go by the method of partitioning like this then we will involve equations for two portion separately. So if we simply adapt the double integration method which we are familiar already then each portion will involve two constants of integration and all together four constants of integration have to be found by substituting the available boundary conditions. Whereas here in macaulay's method the constants reduced to two and we can easily substitute two boundary conditions to get the two constants of the integration. This is the advantage of macaulay's method and more over if we involve more number of loads and definitely we will involve number of portions. So any portion can be partition like this and then leaving the common term we can write the extra term in the other portion and then proceed in the manner which we discussed now.

Now I want slope at the supports, first I want slope at support A, so slope at support A we have the simply supported beams AB subjected to central point load WC. So as we discussed we need to put x equal to zero in the slope equation up to the dotted line. So if you put zero then the slope at A which is represented a,

$$EI.\theta_A = \frac{Wl^2}{16}$$

$$\theta_A = \frac{Wl^2}{16EI}$$

Similarly if I want slope at B, I need to put x equal to l in the entire equation. Because x equal to l lies in the region CB. Thereby we will be getting slope at B ass,

$$EI.\theta_B = -\frac{Wl^2}{4} + \frac{Wl^2}{16} + \frac{Wl^2}{8}$$

$$= \frac{-4Wl^2 + Wl^2 + 2Wl^2}{16} = -\frac{Wl^2}{16}$$

$$\therefore \theta_B = -\frac{Wl^2}{16EI}$$

We know that the deflection is maximum where slope is zero, equating slope to zero we get,

$$-\frac{Wx^2}{4} + \frac{Wl^2}{16} = 0$$

$$\therefore \frac{x^2}{4} = \frac{l^2}{16}$$

$$x^2 = \frac{l^2}{4}$$

$$x = \frac{l}{2}$$

So maximum deflection occur at $x = l/2$ where slope is,

$$\therefore EI.y_{\max} = -\frac{(Wl/2)^3}{12} + \frac{Wl^2}{16} \times \frac{l}{2}$$

$$= -\frac{Wl^3}{96} + \frac{Wl^3}{32}$$

$$= \frac{-2Wl^3 + 6Wl^3}{192} = -\frac{4Wl^3}{192} = \frac{Wl^3}{48}$$

$$\therefore y_{\max} = \frac{Wl^3}{48EI}$$

This is standard case a simply supported beam with central point load is a standard case and slope at supports will be,

$$\theta_A = \theta_B = \pm \frac{Wl^2}{16EI}$$

Deflection will be maximum at the center, So the deflection will be

$$y_c = \frac{Wl^3}{48EI}$$

Example 2:

A simply supported beam AB of span 2.8m carries point load of 60kN at 1m from A. Determine the maximum deflection. Also find the deflection under the load point. Take $EI = 4 \times 10^{12} Nmm^2$.

Solution:

The difference between the earlier problem and this problem was that the earlier one we had central point load and now we have eccentrics load. So in this case how we are find the slope and deflection at different points. Before proceeding we need to find the reaction at supports. In the earlier case the problem being symmetrical found the reaction is on either side.

In case if we have eccentric load the reactions can be found using $Wa/l \& Wb/l$, say we have a simply supported beam and we have load W acting at distance A and B then this reaction will be,

$$R_A = \frac{Wb}{l} = \frac{60 \times 1.8}{2.8} = 38.57kN$$

$$R_B = \frac{Wa}{l} = \frac{60 \times 1}{2.8} = 21.43kN$$

If we don't know the value of this equation also we can find the reactions by using the static equilibrium equation $\varepsilon_v = 0 \& \varepsilon_m = 0$. Now we have two portions which are portion AC and portion CB.

Consider a section in portion AC at distance x, we have reaction A at 38.57kN. So bending moment at xx will be,

$$M_{xx} = 38.57 \times x$$

$$= 38.57x$$

Then portion CB and consider a section xx and let the distance be x. So now we will be requires the distance between the 60kN load and section x which will be equal to (x-1) meters and now the bending moment will be,

$$M_{xx} = 38.57x - 60(x-1)$$

So we should be familiar with finding the bending moment at any section which we are already studied in unit 1. Now to obtain the slope and deflection equation we have the general equation,

Integration equation (1) with respect to x, we get

$$EI.\frac{dy}{dx} = -38.57\frac{x^2}{2} + C_1 \Big| + \frac{60(x-1)^2}{2} - - - (2)$$

Integrating equation (2) with respect to x, we get

$$EI.y = -\frac{38.57x^3}{6} + C_1x + C_2 \Big| + \frac{60(x-1)^3}{6} - - - (3)$$

This is the obtained equation for the slope and the deflection of the simply supported beam.