B. ARCHITECTURE

MECHANICS OF STRUCTURES – 1 (AR6201)

PROPERTIES OF SECTION

Lecture - 8

Movement of inertia of a Rectangular Section:

Let us calculate the moment of inertia of a rectangular section by the method of integration. Let us consider a rectangular section of width b and depth d. Let CG be its centre of gravity, horizontal axis passing through the centre of gravity is XX axis, so which will be centroidal XX axis and the vertical axis passing through CG of the rectangle will be YY axis, which will be centroidal YY axis.

Our aim is to calculate the moment of inertia of this rectangular section about its centroidal XX axis and YY axis, for which we shall consider a small strip area of very less thickness, let the thickness of this trip be dy, dy is the thickness of the strip and let the distance of this particular strip from XX axis be y. So let us name this particular strip as PQ, so dy is thickness of strip PQ and y is the distance of the strip from XX axis.

Moment of inertia of this strip about XX axis will be

$$= \frac{b * dy * y^2}{\text{area}}$$

Moment of inertia of the whole rectangle about XX axis is

$$I_{xx} = \int_{y=-d/2}^{y=d/2} bdy * y^2$$

In order to get the moment of inertia of this whole rectangle we should move this strip from its bottom up to its top, so the y distance from XX axis will vary from –d/2 to +d/2. So I_{xx} for the whole rectangle is obtained by the method of integration

$$I_{xx} = b\int_{y=-d/2}^{y=d/2} y^2 * dy$$

$$I_{xx} = b\left[\frac{y^3}{3}\right]$$

By substituting upper limit and lower limit we get

$$I_{xx} = \frac{b}{3}\left[\left(\frac{d}{2}\right)^3 - \left(-\frac{d}{2}\right)^3\right]$$

$$I_{xx} = \frac{b}{3}\left[\frac{d^3}{8} + \frac{d^3}{8}\right]$$

$$I_{xx} = \frac{b}{3} * \frac{d^3}{4}$$

$$I_{xx} = \frac{bd^3}{12}$$

So the above equation is the moment of inertia about its centroidal XX axis. In the same manner if we calculate the moment of inertia about its centroidal YY axis it is

$$I_{yy} = \frac{db^3}{12}$$

Perpendicular Axis Theorem:

Now let us see perpendicular axis theorem. Statement of theorem will be like this if I_{xx} and I_{yy} are the moment of inertia of a plane section about two mutually perpendicular axis meeting at a point O, then the moment of inertia of this plane section I_{zz} about an axis zz perpendicular to the xy plane and passing through the intersection point O is given by

$I_{zz} = I_{xx} + I_{yy}$

Now considered a lamina figure, now this is a lamina area which is in xy plane, so the x and y axis are meeting at the point O. Also I have shown the z axis which is perpendicular to xy plane and passing through the point O in this figure. Now this laminar figure is in xy plane and if I_{yy} is the moment of inertia of this lamina figure about its centroidal XX axis and if is a moment of inertia of this laminar figure about its YY axis, then we can get the value of moment of inertia of this laminar figure about an zz axis which is perpendicular to xy plane and passing through the intersection point of x and y axis as I_{zz} which is

$I_{ZZ} = I_{xx} + I_{yy}$

So this is the perpendicular axis theorem now we shall see the proof of perpendicular axis theorem.

Proof:

This lamina figure is in xy plane meeting at point O through which the zz axis which is perpendicular xy plane is passing. Now on this lamina figure we shall consider a small area which is the hatched one. Let the distance of the small hatched area from the y axis be x and the distance of the small hatched area from the x axis be y and the distance of the small hatched area from the z axis be r. Let dA be the area of the small hatched lamina

Moment of inertia of this small lamina about XX axis is

$I_{xx} = dA*y^2$

Similarly the moment of inertia of this small lamina about YY axis is

$I_{yy} = dA*x^2$

Add the above two equations

$I_{XX} + I_{yy} = dA(x^2+y^2)$

From the above figure we say it as

$r^2 = x^2 + y^2$

So $I_{xx} + I_{yy}$ becomes

$I_{xx} + I_{yy} = dA*r^2$

$dA*r^2 = I_{zz}$

Hence we have proved that

$I_{ZZ} = I_{xx} + I_{yy}$

Moment of inertia of a Circular Section:

We have a circular section of radius r, within the circular section we consider a small ring, radius of the small ring from the origin be x, thickness of the ring be dx.

Let dA be the area of the ring which is under consideration, dA can be calculated

$$dA = 2\pi dx$$

Now we shall calculate the moment of inertia of the ring about ZZ axis, ZZ axis is the axis perpendicular to XY plane and which passes through the origin O. Moment of inertia of this particular ring is

$I_{zz} = dA*x^2$

Moment of inertia of the circular section about ZZ axis is

$$I_{zz} = \int_{x=0}^{x=r} dAx^2$$

Now in order to cover this entire section this ring has to be moved from the send point upto its circumference point. So the x distance will vary from 0 upto maximum of r and hence the limits come as 0 to r. Substituting the value of dA in above equation we get

$$I_{zz} = \int x^2 * 2\pi x dx$$

$$I_{zz} = \int_{x=0}^{x=r} 2\pi x^3 dx$$

$$I_{zz} = 2\pi \int_{x=0}^{x=r} x^3 dx$$

Integrating we get

$$I_{zz} = 2\pi\left[\frac{x^4}{4}\right]$$

Substituting upper limit and lower limit values we get

$$I_{ZZ} = \frac{2\pi}{4} * \left[r^4 - 0\right]$$

$$I_{zz} = \frac{\pi}{2} * r^4$$

$$I_{zz} = \frac{\pi}{2} * \frac{d^4}{16}$$

I_{zz} for circular section is

$$I_{zz} = \frac{\pi d^4}{32}$$

From perpendicular axis theorem

$I_{zz} = I_{xx} + I_{yy}$

Now we shall get the value of I_{xx} and I_{yy} for the circular section. Now a circular section is a symmetrical section with respect to XX and YY axis. It's a perfect symmetrical section with respect to XX and YY axis, Moment of inertia of the circular section and centroidal YY axis will be same. Therefore

$$I_{xx} = \frac{\pi d^4}{64} = I_{yy}$$

Parallel Axis Theorem:

The statement for the parallel axis theorem will be if the moment of inertia of a plane section about an axis passing through its centre of gravity represented as 'I_g', then the moment of inertia of the section about an axis parallel to the centroidal axis and which is at a distance h from the centroidal axis is given by

$I_{AB} = I_G + ah^2$

Consider a lamina this dot represent the centre of gravity of this lamina figure, let this broken line represent the centroidal axis that is axis passing through the centre of gravity of this laminar figure, if the area of the lamina figure ‘a’. We can calculate the moment of inertia of this laminar figure about reference Axis AB this reference axis is parallel to the centroidal axis and the distance between the reference axis and the centroidal axis is h by parallel axis theorem moment of inertia of this lamina about the reference axis AB will be given by

$I_{AB} = I_G + ah^2$

Where a is the area of section and h is the distance between centroidal axis and the reference axis AB .

Proof:

We shall consider a circular section. The dot represents centroid of the circular section, this is the horizontal centroidal axis, we shall consider the reference axis AB. The distance between the centroidal axis and the reference axis is h. Within the circular section we can consider a small narrow strip, the shaded one is the small narrow strip. Distance between this strip and centroidal axis is y. Let dA be the area of the strip which we have consider.

Moment of inertia of the strip about centroidal axis will be

$=dA*y^2$

Moment of inertia of the whole circular section about the centroidal axis will be

$I_G = \Sigma dAy^2$

Now we need to find out moment of inertia of the circular section about this reference axis AB, Therefore

$I_{AB} = \Sigma dA(y+h)^2$

$I_{AB} = \Sigma dA\ (y^2+h^2+2hy)$

$I_{AB} = \Sigma dAy^2 + \Sigma dAh^2 + \Sigma 2hydA)$

$I_{AB} = I_G + ah^2 + 2h\Sigma ydA)$

Where ΣydA is the algebraic sum of moment of all areas about the centroidal axis which will be zero, because this is the circular section this is the centroidal axis, if we consider the area above and below the XX axis, for the area above XX axis y distance will be positive and for the area below the XX axis y distance will be negative, so summation of areas will cancel out each other so this particular term will be zero. So I_{AB} will become

$I_{AB} = I_G + ah^2$

This the proof for parallel axis theorem.