# Bachelor of Architecture

## Mathematics

## Lecture 7

In this lecture we are going to see the Sphere, Tangent plane, Plane section of a sphere and then finally summary.

### Sphere:

We know the sphere is a three dimensional solid body and the volume of a sphere can be obtained by the formula $\frac{4}{3}\pi r^3$. In analytical geometry a sphere is represented by an equation. For that first we need to define a sphere.

The locus of a point which moves in space in such a way that whose distance form a fixed point in space is always same is called a sphere. The fixed point is known as center of the sphere and the distance of the variable point from the fixed point is known as the radius of the sphere.

The surface generated by z circle when it rotates about its any diameter is also called as a sphere.

### Equation of a sphere:

To obtain the equation of a sphere whose center and radius are given.

Let C be the center of a sphere whose co-ordinates are assumed $(\alpha, \beta, \gamma)$ and let r be the radius of this sphere. Let $p(x, y, z)$ be any variable on the surface of the sphere whose equation to be determined as shown in the following figure.

Let us imagine a circle, in that circle C be the center and P be the point on the circle. Then let $cp = r$ ,

$$cp^2 = r^2 ---------(1)$$

Since CP is the distance between two points $c(\alpha, \beta, \gamma)$ & $p(x, y, z)$. Then

$$cp = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}$$

Now substitute this value of CP in equation (1) we get

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2$$

This is the required equation of a sphere.

The equation of a sphere whose end points of a diameter are given is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

In this case the center of this sphere is $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$.

General form of a sphere:

The equation of a sphere of radius r having center $(\alpha,\beta,\gamma)$ is given by

$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2$ or

$$x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z+\alpha^2+\beta^2+\gamma^2-r^2=0------(1)$$

From equation (1) we observed that the equation (1) is a second degree in $x,y,z$ having no terms of $xy,yz,zx$ and co-efficient of $x^2,y^2,z^2$ equal to 1.Therefore the equation of the type

$$x^2+y^2+z^2+2ux+2uy+2wz+d=0---------(2)$$

Having the same character as (1) has is called the general equation of a sphere. Now comparing equation (2) with (1) we get $\alpha=-u,\beta=-v,\gamma=-w$ and $d=\alpha^2+\beta^2+\gamma^2-r^2$. Thus the center and the radius of (2) are given by $(-u,-v,-w)$ and $\sqrt{u^2+v^2+w^2}-d$ respectively.

**<u>Example 1:</u>**

Find the equation of the sphere whose center is $(2,-3,4)$ and radius $\sqrt{51}$.

**<u>Solution:</u>**

The general form of the equation of the sphere is,

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2$$

Here in this problem the center is $(2,-3,4)$ and radius $\sqrt{51}$ so substituting this values in the above equation we get,

$$(x-2)^2+(y-3)^2+(z-4)^2=51$$

So simplifying this we get,

$$x^2+y^2+z^2-4x+6y-8z-22=0$$

**Example 2:**

Find the equation of a sphere passing through origin and making intercept $a,b,c$ with the axes respectively.

**Solution:**

Let the equation of the sphere be,

$$x^2+y^2+z^2+2ux+2uy+2wz+d=0--------(1)$$

According to the problem the sphere passes through the origin and the co-ordinates of the origin is $(0,0,0)$.And if the sphere passes through the origin then d value will become 0.The sphere also cut the axis at $A,B,C$. Then $OA=a,OB=b,OC=c$ thus the point $A,B,C$ are coordinately given as $(a,0,0);(0,b,0)\,\&\,(0,0,c)$.

Now the general equation of the sphere also passes through this three points. So will get the equation as,

$$a^2+2ua+d=0$$

$$2u=-a$$

Similarly,

$$2v=-b$$

$$2w=-c$$

Now substituting this values of $u,v,w$ we get,

$$x^2+y^2+z^2-ax-by-cz=0$$

It is the required equation of a sphere which has the origin and cuts the co-ordinate axis $A,B,C$.

**Tangent Plane:**

Tangent plane is the one which touches the sphere at only one point. Let us get the equation of the tangent plane to a sphere $x^2+y^2+z^2+2ux+2uy+2wz+d=0$ at the point $(x,y,z)$ on the sphere. Let the equation of a straight line through the point $(x, y, z)$ and having the direction cosines $lm, n$ be,

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}------(1)$$

And let the equation of the sphere be,

$$x^2+y^2+z^2+2ux+2uy+2wz+d=0--------(2)$$

The point of intersection of the line (1) with the given sphere (2) are given by,

$$r^2+2r[l(u+x_1)+m(v+y_1)+n(w+z_1)]+(x_1)^2+(y_1)^2+(z_1)^2+2ux_1+2vy_1+2wz_1+d=0---(3)$$

Since the point $(x, y, z)$ is on the sphere (2) then,

$$x_1^2+y_1^2+z_1^2+2ux_1+2yv_1+2wz_1+d=0$$

Now equation (3) becomes,

$$r^2+2r[l(u+x_1)+m(v+y_1)+n(w+z_1)]=0------(4)$$

This equation (4) has one value of r is zero. Since r is the distance of the points of intersection of line with the sphere from the point $(x_1, y_1, z_1)$. But r is obtained equal to zero. Thus the point of intersection coincide with the point $(x_1, y_1, z_1)$. Hence the line (1) become a tangent line of the sphere (2).

Therefore other point of intersection must also coincide with $(x_1, y_1, z_1)$ for which other root of the equation (4) must be zero. That is the condition for this is given by

$$l(u+x_1)+m(v+y_1)+n(w+z_1)=0--------(5)$$

Hence we can say that if the direction cosines of the line (1) satisfy the condition (5) then this line is a tangent line. Therefore the tangent plane at $(x, y, z)$ is the locus of such tangent lines for all values of $(l, m, n)$.

Now eliminating $(l, m, n)$ from (1) and (5) we get

$$(x-x_1)(u+x_1)+(y-y_1)(v+y_1)+(z-z_1)(w+z_1)=0 \text{ Or}$$

$$xx_1+yy_1+zz_1+ux+vy+wz-\{(x_1)^2+(y_1)^2+(z_1)^2+ux_1+vy_1+wz_1\}=0----(6)$$

Since $(x_1)^2+(y_1)^2+(z_1)^2+2ux_1+2vy_1+2wz_1+d=0$ the above equation become,

$xx_1+yy_1+zz_1+ux+vy+wz+ux_1+vy_1+wz_1+d=0$ Or

$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$

It is the required equation of a tangent plane at $(x_1,y_1,z_1)$ to the sphere $x_1^2+y_1^2+z_1^2+2ux+2yv+2wz+d=0$.

**The plane section of a sphere:**

When a plane cut the sphere a cross section of a sphere is obtained as circle. Now prove that the cross section of a sphere cuts by a plane is a circle and find also its radius and center. Let the equation of the sphere and the plane be,

$x^2+y^2+z^2+2ux+2yv+2wz+d=0------(1)$

$ax+by+cz+d_1=0--------(2)$

In the figure the dotted line represents the cross section of a sphere cuts off by a plane (2). Let C be the center of the sphere (1) draw a perpendicular from C to the plane (2) which meets the plane at C'. Join C' to P where P is on the surface of the sphere and as the dotted line. Then C'P lies in the plane and CC' is perpendicular to the plane. Thus CC is perpendicular to every line which is in the plane. Therefore $\angle CC'P=90^0$.

In the triangle CC'P, $\angle CC'P=90^0$ then $C'P^2=cp^2-CC'2orC'P=\sqrt{cp^2-CC'2}$. Since CP is the radius of the sphere and CC' is the length of the perpendicular drawn from C to the given plane. Consequently the equation (1) and (2) simultaneously give the equation of a circle.

**Summary:**

So from this lecture we have discussed about the sphere, the locus of a point which moves in space such that whose distance from a fixed point in space is always same is called sphere. Then the general equation of the sphere is $x^2+y^2+z^2+2ux+2yv+2wz+d=0$. And $xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$ be the equation of the tangent plane to the sphere. The plane section of the sphere is a circle.

After listening to this lecture you can answer the following questions.

**Questions:**

1. Find the equation of the sphere on the join $(2,-3,1)$ and $(3,-1,2)$ as a diameter.

2. Find the center and radius of the sphere $2x^2 + 2y^2 + 2z^2 - 2x + 4y - 6z = 15$.
3. Give the equation of a tangent plane.