Bachelor of Architecture

Mathematics

Lecture 6

In this lecture we are going to see the three dimensional analytical geometry. In the three dimensional analytical geometry we are going to see the Co-planer lines, Shortest distance between skew lines and finally the summary.

Co-planer lines:

We know two lines or a set of lines that lie on the same plane is called co-planer lines.Here both the lines may be in a symmetrical form or only one line can be symmetrical form and other will be in non-symmetrical form. And both lines may be in the non-symmetrical form. First we will see the both lines in a symmetrical form.

When both lines are in symmetric form:

To obtain the condition that two lines may intersect and the equation of the plane in which they lie. Let the equation of two lines be given by,

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1}----------(1)$$

$$\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}----------(2)$$

The equation of a plane through the line (1) is

$$(x-\alpha_1)+b(y-\beta_1)+c(z-\gamma_1)=0-------(3)$$

Where $al_1+bm_1+cn_1=0-------(4)$

If the line (2) lies in the same plane then the normal of this plane is perpendicular to (2), so we have

$$al_2+bm_2+cn_2=0-------(5)$$

Further since the point $(\alpha_2,\beta_2,\gamma_2)$ also lie on the plane then,

$$a(\alpha_2-\alpha_1)+b(\beta_2-\beta_1)+c(\gamma_2-\gamma_1)=0----------(6)$$

Now eliminating a,b,c from (4),(5) and (6) we get the required condition

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

And hence the required plane is

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

<u>When one line is symmetrical form and other in general form:</u>

To obtain the 1condition that the lines,

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

Here, $xa_1+yb_1+zc_1=0$; $xa_2+yb_2+zc_2=0$

These two lines are the co-planer. Any point on the first line is $(lr+\alpha, mr+\beta, nr+\gamma)$. This point will lie on the second line. Then we have

$$a_1(lr+\alpha)+b_1(mr+\beta)+c_1(nr+\gamma)+d_1=0$$

$$r=-\frac{a_1\alpha+b_1\beta+c_1\gamma+d_1}{a_1l+b_1m+c_1n}$$

Equating the values of r, we get the required condition as,

$$\frac{a_1\alpha+b_1\beta+c_1\gamma+d_1}{a_1l+b_1m+c_1n}=\frac{a_2\alpha+b_2\beta+c_2\gamma+d_2}{a_2l+b_2m+c_2n}$$

<u>Remark</u>: In the above condition both the denominator should not be equal to zero.

<u>When both the lines are in general form:</u>

To obtain the condition the lines may intersect. When the lines intersect we will get the equation of co-planer as, $xa_1 + yb_1 + zc_1 + d_1 = 0$

$$xa_2 + yb_2 + zc_2 + d_2 = 0 ----------(1)$$

$$xa_3 + yb_3 + zc_3 + d_3 = 0$$

$$xa_4 + yb_4 + zc_4 + d_4 = 0 ----------(2)$$

Now eliminating x, y, z from equation (1) and (2) we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

This is the required condition.

Example 1:

Show that the lines $\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$ and $\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ are co-planer and find the equation of the plane containing them.

Solution:

The equation of the plane which contain the first line and parallel to the second line is given by,

$$\begin{vmatrix} x+3 & y+5 & z-7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0$$

If we simplify this determinant we get,

$$(x+3)(-3+15) - (y+5)(-2+12) + (z-7)(10-12) = 0$$

$$(x+3)(12) - (y+5)(10) + (z-7)(-2) = 0$$

$$12x + 36 - 10y - 50 - 2z + 14 = 0$$

$$12x - 10y - 2z - 50 + 50 = 0$$

$$12x - 10y - 2z = 0$$

$6x-5y-z=0$

This is the equation of plane which contains $\frac{x+3}{2}=\frac{y+5}{3}=\frac{z-7}{-3}$ and parallel to the line $\frac{x+1}{4}=\frac{y+1}{5}=\frac{z+1}{-1}$. This plane passes through $(-1,-1,-1)$. Hence these two lines are co-planer.

Shortest distance between skew lines:

Shortest distance between the lines can only have the meaning, if the lines do not intersect or the lines are not lying in the same plane. Lines which have this character are called skew lines. Then the length between them which is perpendicular to both skew lines is called shortest distance.

This shortest distance between the skew lines is obtained

1. When both the lines in symmetric form,
2. When one line in symmetric form and other in general form,
3. When both lines are in general form.

Now let us see the first case.

When both lines in symmetric form:

In this case two ways are there to find the shortest distance between two lines. They are projection method and the co-ordinates method.

Projection Method:

In projection method the length of the shortest distance is given by

$$l(\alpha_1-\alpha_2)+m(\beta_1-\beta_2)+n(\gamma_1-\gamma_2)$$

Here l,m,n are direction cosines of a shortest distance. And $p(\alpha_1,\beta_1,\gamma_1) and Q(\alpha_2,\beta_2,\gamma_2)$ are the points in the first line and second line.

Example 1:

Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4},\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$

Solution:

Let l, m, n be the direction cosine of the shortest distance. Then the shortest distance is perpendicular to both the lines therefore we have

$$2l + 3m + 4n = 0 ---------(1)$$

$$3l + 4m + 5n = 0 ---------(2)$$

Then solving this two equations we get

$$\frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9}$$

$$\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$$

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

$$\therefore l = \frac{1}{\sqrt{6}}; m = \frac{2}{\sqrt{6}}; n = \frac{1}{\sqrt{6}}$$

These l, m, n are the direction cosine of the shortest distance. Then the point P which has co-ordinates $P(1,2,3)$ and the point Q has the co-ordinates $Q(2,4,5)$ are the points on the given lines. So the projection of PQ on the shortest distance is given by

$$S.D = (2-1)\frac{1}{\sqrt{6}} + (4-2)(-\frac{2}{\sqrt{6}}) + (5-3)\frac{1}{\sqrt{6}} = \frac{-1}{\sqrt{6}}$$

Let us take only the magnitude, so the shortest distance for the given line is $= \frac{1}{\sqrt{6}}$.

Summary:

So from this lecture we have learned that the co-planar are the lines which lies on the same plane. Conditions satisfied by the lines to be co-planar is obtained as

- When both line in symmetrical form
- When one line in symmetrical form and other in general form
- When both lines are in general form.

The shortest distance between the skew lines is obtained as $l(\alpha_1 - \alpha_2) + m(\beta_1 - \beta_2) + n(\gamma_1 - \gamma_2)$. After listening to this lecture you can answer the following questions.

Questions:

1. What is known as the co-planar lines?
2. Prove that the lines $\frac{x}{1}=\frac{y-2}{2}=\frac{z+3}{3}$ and $\frac{x-2}{2}=\frac{y-6}{3}=\frac{z-3}{4}$ are co-planar.
3. Find the length and equations of shortest distance between $3x-9y+5z=0=x+y-z$ and $6x+8y+3z-13=0=x+2y+z-3$.