Bachelor of Architecture

Mathematics

Lecture 5

In this lecture we are going tosee the three dimensional analytical geometry. Inthis three dimensional analytical geometry will see Equation of a plane and the Equation of a straight line and the finally summary.

Equation of a Plane:

We know that the plane is the two dimensional object. So every equation of first degree in x,y,z represents a plane. The general equation of the first degree is given by

$$ax+by+cz+d=0 ----------(1)$$

Where a,b,c,d are the real constants and at least one if the a,b,c is non-zero. Let $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ be any two points on the surface given by equation (1), then

$$ax_1+by_1+cz_1+d=0 ---------(2)$$

$$ax_2+by_2+cz_2+d=0 ---------(3)$$

Multiplying equation (3) by λ and adding to equation (2) we have,

$$a(x_1+\lambda x_2)+b(y_1+\lambda y_2)+c(z_1+\lambda z_2)+d(1+\lambda)=0$$

Then divided by $(1+\lambda)$ we have

$$a\left(\frac{x_1+\lambda x_2}{1+\lambda}\right)+b\left(\frac{y_1+\lambda y_2}{1+\lambda}\right)+c\left(\frac{z_1+\lambda z_2}{1+\lambda}\right)+d=0 ----------(4)$$

From equation (4) we observe that the point,

$$\left(\frac{x_1+\lambda x_2}{1+\lambda}\right),\left(\frac{y_1+\lambda y_2}{1+\lambda}\right),\left(\frac{z_1+\lambda z_2}{1+\lambda}\right)$$

Satisfy equation (1) and this point is on the straight line joining P and Q. Since λ is a parameter so it can take any real value except 1. Hence all the points on the line PQ are satisfied by equation (1) consequently. Then, $ax+by+cz+d=0$

This represents a plane in general form. So in this discussion you learned that the equation of the plane containing the point P and Q with the co-ordinates $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Note:

In this equation $ax + by + cz + d = 0$ we can note that the co-efficientsa,b,c represents the direction ratios of a normal to this plane.

Corollary 1:

General equation of a plane passing through a given point, let $A(x_1, y_1, z_1)$ be a given point and let the equation of a plane be $ax + by + cz + d = 0$ that passes through $A(x_1, y_1, z_1)$ then we have, $ax_1 + by_1 + cz_1 + d = 0$

So from this corollary we have learned that if any plane passes through a point then combining the point and the plane will get an equation. Then subtracting this two equations we get,

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

This is the required general equation of a plane passing through (x_1, y_1, z_1).

Corollary 2:

Equation of a plane that passes through the origin is given by

$$ax + by + cz = 0$$

Corollary 3:

Equation of the plane parallel to the co-ordinate axes are given by

1. If a=0 then $by + cz + d = 0$ is a plane parallel to the x-axis.
2. If b=0 then $ax + cz + d = 0$ is a plane parallel to the y-axis.
3. If c=0 then $ax + by + d = 0$ is plane parallel to the z-axis.

Intercept form of a plane:

The equation of a plane which cuts the co-ordinates axes is called intercept form of a plane. The equation of intercept form of the plane is give as, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Normal form of a plane:

The equation of a plane in terms of a perpendicular distance from origin to the plane and direction cosines of this perpendicular is called normal form. The equation of the normal form of a plane is $lx+my+nz=P$

Equation of a plane passing through three points,

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

It is the required equation of a plane passing through three points.

Example 1:

A plane meets the co-ordinates axes in A, B, C such that the centroid of the triangle ABC is the point (p,q,r). Show that the equation of the plane is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3.$

Solution:

Let the equation of the plane be,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 ------- (1)$$

The plane (1) meets the co-ordinate axes in $A(a,0,0), B(0,b,0) and C(0,0,c)$. Therefore the co-ordinates of the centroid of the ΔABC is

$$\left(\frac{a+0+0}{3},\frac{0+b+0}{3},\frac{0+0+c}{3}\right) or \left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$$

Since centroid of ΔABC is given as (p,q,r) where,

$$p=\frac{a}{3} or a=3p$$

Similarly, $b=3q, c=3r$

Now substituting the values of a,b,c in equation (1) we get,

$$\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}=1$$

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$$

This is the required equation to be proved as given in the problem.

Example 2:

Find the equation of the plane passing through the point $p(2,3,-1)$ and perpendicular to OP, where O is the origin.

Solution:

First we need to get the equation of the plane passing through the point $P(x_1, y_1, z_1)$ will be,

$$a(x-x_1)+b(y-y_1)+c(z-z_1)$$

Now substitute the value of (x_1, y_1, z_1) we get,

$$a(x-2)+b(y-3)+c(z-1)=0------(1)$$

According to the problem this equation is perpendicular to OP. Now the directional ratio of OP are $(2-0),(3-0),(-1-0)$. So it will be $(2,3,-1)$. This directional ratio is proportional to a,b,c therefore we will get,

$$\therefore a=2k, b=3k, c=-k$$

Where in this case k should not be equal to zero and the value of k must be the real one. Now substitute the values of a,b,c in equation (1) we get,

$$2k(x-2)+3k(y-3)-k(z+1)=0$$

Now simplifying this equation we get,

$$2(x-2)+3(y-3)-(z+1)=0$$

Since the value of k is not equal to zero. Now simplifying the above equation further will get,

$$2x+3y-z-14=0$$

This is the required equation of the plane which passes through the point and perpendicular to OP a line drawn from the origin.

Equations of a line:

Equations of a line can be obtained in different form. Among that symmetrical form and the non-symmetrical form are the two major categories.

Symmetrical form:

To obtain the equation of the straight line which is passing through a given point (α, β, γ) and having the direction cosine l, m, n.

Let OX, OY, OZ be the rectangle axes with region O and let A be a point whose co-ordinates are (α, β, γ) as shown in the following.Where this co-ordinate system there will be a X-axis, Y-axis and the Z-axis and O is the origin. A point A is assumed where the co-ordinates of the points are $A(\alpha, \beta, \gamma)$. Another point P is assumed where the co-ordinates of the point be $P(x, y, z)$.

Let $P(x, y, z)$ be any point on this line having the direction cosine l, m, n. Therefore the direction ratios of the line AP are $(x-\alpha, y-\beta, z-\gamma)$. The direction cosine of the line AP are given as l, m, n.

$$l = \frac{x-\alpha}{AP}; m = \frac{y-\beta}{AP}; n = \frac{z-\gamma}{AP} or$$

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = AP$$

This is the symmetrical form of a line.

Corollary 1:

The equation of a line passing through a point (α, β, γ) having direction ratios a, b, c then its symmetrical form is

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$$

Corollary 2:

The equation of a line in symmetrical form passing through two points $(x_1, y_1, z_1) and (x_2, y_2, z_2)$ is

$$\frac{x-X_1}{X_2-x_1} = \frac{y-Y_1}{Y_2-y_1} = \frac{z-Z_1}{Z_2-z_1}$$

Non-Symmetrical form:

A straight line is the intersecting of two planes. Therefore the equations of two planes simultaneously give the equation of a straight line in non-symmetric form.

$$xa_1 + yb_1 + zc_1 + d_1 = 0$$

$$xa_2 + yb_2 + zc_2 + d_2 = 0$$

Summary:

So in this lecture we learned that every equation of first degree in x, y, z represents a plane. The equation of the intercept form of the plane is given by $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = 1$.

The equation of the normal form of a plane is $lx + my + nz = P$. And we learned the equation of a line in the symmetrical form and non-symmetrical form.

After listening to this lecture you can answer the following questions.

Questions:

1. Give the equation of a plane in intercept form and normal form.
2. Find the equation of a plane passing through points $(0,-1,0), (2,1,-1), (1,1,1)$.
3. Find the equations of a plane passing through $(3,4,5) and (2,3,4)$.