Bachelor of Architecture

Mathematics

Lecture 4

In this lecture we are going to discuss the Three dimensional analytical geometry. In that three dimensional analytical geometry we are going to see the Direction cosine and ratios and the Angle between two lines and then finally summary.

Direction Cosine:

In order to understand the direction cosine we need to understand some terms in analytical geometry. The first one is,

Skew lines:

Two non-intersecting lines that is non-co-planer lines are called skew lines.

Example:

Any two opposite edges of a tetrahedron are skew lines.

Angle between two lines:

If the lines are co-planer that is intersecting then the acute angle between them gives the angle between the lines. If the lines are skew then the angle between two lines drawn parallel to the given lines through an arbitrarily chosen point in the space gives the angle between the given lines.

Using these two concepts we can go to the direction cosines of a line.

Direction cosine of a line:

If a line makes angles α, β, γ with the positive directions of the x,y and z axis respectively then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction of cosines of the given line and angle α, β, γ are called direction angles of the given line.

Direction cosines of a line are generally denoted by l, m, n and written as $[l, m, n]$.

$l = \cos\alpha, m = \cos\beta, n = \cos\gamma$

Example 1:

Find the direction cosine of the co-ordinates axes.

Solution:

By this example we will learn how to find the direction cosine of a line. Here in this problem we are asked to find the direction cosine of the co-ordinate axis. So let us take the X-axis, this X-axis make an angle 0^0 with itself. Then the X-axis makes an angle 90^0 with the Y-axis. Then this X-axis again makes an angle 90^0 with the Z-axis.

So the direction cosine of this line will be,

$$[Cos0^0, Cos90^0, Cos90^0]$$

$$[1,0,0]$$

Direction cosine of X-axis is $[1,0,0]$. Then the Y-axis we know that Y-axis makes an angle 90^0 with X-axis and an angle 0^0 with itself. Then the same Y-axis makes an angle 90^0 with the Z-axis. So the direction cosine of the Y-axis will be

$$[Cos90^0, Cos0^0, Cos90^0]$$

$$[0,1,0]$$

The Direction cosine of the Y-axis is $[0,1,0]$.Finally we are going to find the direction cosine of the Z-axis. These Z-axis makes an angle 90^0 with the X-axis and 90^0 with the Y-axis and 0^0 with itself. Hence the direction cosine of the Z-axis will be,

$$[Cos90^0, Cos90^0, Cos0^0]$$

$$[0,0,1]$$

So the direction cosine of the Z-axis is $[0,0,1]$. Now the direction cosine of all the co-ordinate axis are $[1,0,0]$, $[0,1,0]$, $[0,0,1]$. This is how we can find the direction cosine of all the co-ordinate axis.

Relation between direct cosines:

The relation between direct cosines is given as,

$Cos^2\alpha + Cos^2\beta + Cos^2\gamma$

Proof:

To get the proof of this relation we need to get a co-ordinate system. In that co-ordinate system we have X-axis, Y-axis and the Z-axis.The line found for the direction cosine is AB.

According to the concept of finding the cosine of the function we need to draw a line from the origin and it is denoted by OC. This line makes an angle α with X-axis, and makes an angle β with Y-axis and then makes an angle γ with the Z-axis. So the direction cosine of the given line be, $Cos\alpha, Cos\beta, Cos\gamma$

Now let us imagine a point P on the line OC. So co-ordinate of this point be (x_1, y_1, z_1). The distance of the point P from the origin is assumed as r. Now draw a perpendicular line PM and M to a XY plane. The directional cosine to this function can be obtained as,

$$Cos\alpha = \frac{x_1}{r}$$

This direction cosine is obtained from the triangle ΔONP, Similarly we can get the direction cosines,

$$Cos\beta = \frac{y_1}{r}$$

$$Cos\gamma = \frac{z_1}{r}$$

These three directional cosines are obtained from the triangle ΔONP. Now to get the relation among the cosine let us square all these three direction cosine and add them. So on squaring,

$$Cos^2\alpha = \frac{x^2{}_1}{r^2}$$

$$Cos^2\beta = \frac{y^2{}_1}{r^2}$$

$$Cos^2\gamma = \frac{z^2{}_1}{r^2}$$

So summing all this we get,

$$Cos^2\alpha + Cos^2\beta + Cos^2\gamma$$

$$=\frac{x_1^2}{r^2}+\frac{y_1^2}{r^2}+\frac{z_1^2}{r^2}\ =\frac{x_1^2+y_1^2+z_1^2}{r^2}$$

We know in vector the position vector is given by $r^2=x_1^2+y_1^2+z_1^2$. Substituting this value the above expression will become, $=\frac{r^2}{r^2}\ =1$

Example 2:

Find the direction cosines of a line that makes equal angles with the axes.

Solution:

In direction cosine we learned that α is the angle made by the X-axis, β is the angle made by the Y-axis and γ is the angle made by the Z-axis. But here in this problem we are asked to find the line that makes equal angles with the axes.

Let the direction cosine of that line be,

$$[Cos\alpha, Cos\beta, Cos\gamma]$$

$$[Cos^2\alpha, Cos^2\beta, Cos^2\gamma]$$

According to this problem $\alpha=\beta=\gamma$. So substituting this we will get,

$$Cos^2\alpha+Cos^2\alpha+Cos^2\alpha\ =3Cos^2\alpha\ \ =1$$

$$3Cos^2\alpha=1$$

$$Cos^2\alpha=\frac{1}{3};Cos\alpha=\pm\sqrt{\frac{1}{3}}\ =\pm\frac{1}{\sqrt{3}}$$

Then the value of other angles also be same because here the line makes equal angles of the co-ordinate axis. Hence the direction cosine for this problem will be,

$$\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$$

The next one will be, $\left[-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right]$

Here in this problem we learned how to find the direction cosine if the line makes an equal angle with all the three co-ordinate axis.

Direction ratios:

If the direction cosines of a line are proportional to three numbers a,b and c respectively then a,b and c are called direction ratios in short form direction ratios of the line. Thus if [l,m,n] are the actual direction cosine and a,b,c are direction ratios of a line then

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$$

Where l,m,n are the directional cosines and the a,b,c are the numbers to which the directional cosine is proportional which is equal to,

$$\frac{\pm\sqrt{(l^2+m^2+n^2)}}{\sqrt{a^2+b^2+c^2}}$$

$$=\frac{\pm 1}{\sqrt{a^2+b^2+c^2}}$$

Here the value 1 came because the sum of square of the direction cosine is one.

$$\therefore l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}}$$

The direction cosine of the line whose direction are direction ratio a,b,c are $\left[\frac{\pm a}{\sqrt{\sum a^2}},\frac{\pm b}{\sqrt{\sum a^2}},\frac{\pm c}{\sqrt{\sum a^2}}\right]$.

We know that $a^2+b^2+c^2\neq 1$from the direction ratio of the point $P(x_1,y_1,z_1)$ and the point $Q(x_2,y_2,z_2)$.So the direction ratio of the point are

$$(x_2-x_1),(y_2-y_1),(z_2-z_1)$$

Summary:

In this lecture we have learned the Definition of skew lines and angle between two lines and then the Direction cosines and ratios of a line. Then the Angle between two lines have been derived.After this listening to this lecture you can answer the following questions.

Questions:

1. Define skew line.
2. What is known as directional cosine of a line?

3. Define direction ratios or direction number.