Bachelor of Architecture

Mathematics

Lecture 13

In this lecture we are going to see Cauchy-Euler homogeneous linear differential equations, Equation reducible to linear differential equation with constant co-efficient, and Legendre homogeneous linear differential equationsand then summary.

Cauchy-Euler homogeneous linear differential equations:

A linear differential equation of the form,

$$a_0x^n \frac{d^n y}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{dn^{-2}y}{dx^{n-2}} + ... + a_ny = F(x) ----(1)$$

Where $a_1, a_2 ......a_n$ are constants and F is either a constant or a function of X only is called Cauchy-Euler homogeneous linear differential equation. In this equation the index of x and order of derivative is same in each term of the equation. Using the symbols,

$$D(=\frac{d}{dx}); D^2(=\frac{d^2}{dx^2}).........D^n(=\frac{d^n}{dx^n})$$

The equation (1) become,

$$\left(a_0x^nD^n + a_1x^{n-1}D^{n-1} + a_2x^{n-2}D^{n-2} + ...a_n\right)y = F(x)$$

The above equation can be reduced to linear differential equation with constant co-efficient by substituting.

Equation reducible to linear differential equation with constant co-efficient:

To get the equation reducible to linear differential equation with constant co-efficient we need to do some substitution,

$$x = e^z$$

$$\ln x = z$$

So that,

$$\frac{dz}{dx} = \frac{1}{x} ------(3)$$

Now using chain rule for differentiation we obtain,

$$\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}.\frac{dy}{dz}$$

Then designing a new operator for $\frac{d}{dz}$,

$$\frac{d}{dz} = D_1$$

$$x\frac{dy}{dx} = \frac{dy}{dz} = xD_y = D_1 y$$

So this is the way we can reduce the differential equation with constant co-efficient. Similarly for the second derivative we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

The further simplification of the above equation will give,

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$x^2D^2y = D_1(D_1 - 1)y$$

Similarly for the third derivative we can write as,

$$x^3D^3y = D_1(D_1 - 1)(D_1 - 2)y$$

So like this we can write the operator form for fourth derivatives etc., in general it will be,

$$x^nD^ny = D_1(D_1 - 1)(D_1 - 2)........(D_1 - n + 1)y$$

Now substituting the values of xD_y, x^2D^2y, x^3D^3y in equation (1) we get,

$$[a_0D_1(D_1 - 1).......(D_1 - n + 1) + ... + a_{n-2}D_1(D_1 - 1) + ...a_{n-1}D_1 + a_n]y = F(e^z) - -(4)$$

This equation is linear differential equation with constant co-efficient. This equation can be solved by various methods. Now we will apply this concept in problem.

Example 1:

Solve the differential equation $x^2D^2 + xD - 4)y = 0$

Solution:

First write the given equation in operator form,

$$x = e^z$$

$$\ln x = z$$

$$xD = D_1, x^2D^2 = D_1(D_1 - 1)$$

The given equation will be,

$$[D_1(D_1 - 1) + D_1 - 4]y = 0$$

$$[D_1^2 - D_1 + D_1 - 4]y = 0$$

$$[D_1^2 - 4]y = 0$$

This is similar to the second order differential equation then,

$$m = 2, -2$$

$$y = c_1e^{2z} + c_2e^{-2z}$$

$$z = \log x$$

$$y = c_1e^{2\log x} + c_2e^{-2\log x}$$

$$y = c_1x^2 + c_2x^{-2}$$

This is the required solution of the given linear differential equation.

Example 2:

Find the general solution of the differential equation $(x^2D^2 + y)y = 3x^2$

Solution:

Let us start with the method of substitution, we have

$$x = e^z$$

$$(D_1(D_1 - 1) + 1)y = 3e^{2z}$$

$$(D_1^2 - D_1 + 1)y = 3e^{2z}$$

The characteristic equation of the above equation will be,

$$m^2 - m + 1 = 0$$

$$m = \frac{(1 \pm i\sqrt{3})}{2}$$

Then the complementary function of the equation will be,

$$C.F = e^{z/2}[c_1 \cos\left(\frac{2\sqrt{3}}{2}\right) + c_2 \sin\left(\frac{2\sqrt{3}}{2}\right)$$

This can further be simplified by substituting $z = \ln x$.

$$C.F = \sqrt{x}\left[c_1 \cos\left(\ln \frac{x\sqrt{3}}{2}\right) + c_2 \sin\left(\ln \frac{x\sqrt{3}}{2}\right)\right]$$

Now let us go to the particular integral,

$$P.I = \frac{1}{D_1^2 - D_1 + 1} 3e^{2z}$$

Substituting 2 in the place of D we get,

$$= \frac{1}{2^2 - 2 + 1} 3e^{2z} = e^{2z}$$

$$y = \sqrt{x}\left[c_1 \cos\left(\frac{\ln x\sqrt{3}}{2}\right) + c_1 \sin\left(\frac{\ln x\sqrt{3}}{2}\right)\right] + x^2$$

This is the required solution of the given differential equation.

Legendre's homogeneous linear differential equations:

A linear differential equation of the form,

$$\left[(a+bx)^n a_0 D^n + (a+bx)^{n-1} a_1 D^{n-1} + (a+bx)^{n-2} a_2 D^{n-2} + ... + a_n\right] y = F(x) --(1)$$

Where $a, b, a_1, a_2, ...a_n$ are constants and F is either a constants or a function of x only is called Legendre's homogeneous linear differential equation.

Here the index of (a+bx) and the order of derivative is same in each term of such equations. To solve the equation (1) we introduce a new independent variables z such that

Summary:

In this lecture we learnt that the Cauchy-Euler homogeneous linear differential equation and the Cauchy-Euler homogeneous linear differential equation can be reduced to constant co-efficient linear differential equation by substituting $x = e^z$. Solution of this homogeneous linear differential equation consists of complementary function and particular integral.

After listening to this lecture you can answer the following questions.

Questions:

1. Solve $(x^3 D^3 + 3x^2 D^2 - 2xD + 2)y = 0$.
2. Write Legendre's homogeneous linear differential equation.
3. Solve $x^3 y'' + xy' - y = 3x^4$.