Bachelor of Architecture

Mathematics

Lecture 12

In this lecture we are going to see Simultaneous Linear differential equation, Solution by method of elimination, method of differentiation and then finally Summary.

Simultaneous Linear differential equation:

Consider a system of simultaneous linear differential equations which contains a single independent variable and two or more dependent variable. Let x and y be the dependent and where 't' be the independent variable. Thus in such equations there occurs differential co-efficient of x, y with respect to t. Then the Linear differential equation have two types if the right hand side of the equation is zero then it is called homogeneous linear differential equation. If the right hand side has some value then it is non-homogeneous linear differential equation.

Let $D = \frac{d}{dt}$ then such equation can be put into the form,

$$f_1(D)x + f_2(D)y = T_1 -------(1)$$

$$g_1(D)x + g_2(D)y = T_2 -------(2)$$

Where $T_1 \& T_2$ are functions of the independent variable 't' and $f_1(D), f_2(D), g_1(D), g_2(D)$ are all rational integral functions of D with constant co-efficient.

In general the number of equation will be equal to the number of dependent variables. i.e., if there are n dependent variables there will be n equations.

Method of elimination:

In order to eliminate y between equations (1) and (2) operating on both sides of (1) by $g_2(D)$ and on both side of (2) by $f_2(D)$ are subtracting we get

$$\left[f_1(D)g_2(D)-g_1(D)f_2(D)\right]x=g_2(D)T_1-f_2(D)T_2 ------(3)$$

This is a linear differential equation with constant co-efficient in 'x' and 't' and can be solved to give the value of x in terms of 't'. Substituting this value of 'x' in either (1) or (2) we get the value of y in terms of 't'.

Remark 1:

The above equation (1) and (2) can be solved by first eliminating x between them and solving the resulting equation to get y in terms of 't'. Then substituting the y in either (1) or (2) we get the value of 'x' in terms of 't'. This is the method of elimination.

Remark 2:

In the general solution of (1) and (2) the number of arbitrary constants will be equal to the sum of the order of the equation (1) and (2).

Now we will see an example to understand this more clear.

Example 1:

Solve the simultaneous linear differential equation $\frac{dx}{dt}-7x+y=0, \frac{dy}{dt}-2x-5y=0$

Solution:

Let us first express the given equation in operator form that is $D\rightarrow\frac{d}{dt}$. So the given first equation will become,

$$(D-7)x+y=0$$

Then the second equation will become,

$$(D-5)y-2x=0$$

$$(D-7)x+y=0------(1)$$

$$-2x+(D-5)y=0---(2)$$

Now we need to eliminate the x by multiplying the equation (1) by (2) and operating (D-5), we get

$$2(D-7)x+2y=0------(3)$$

$$-2(D-7)x+(D-7)(D-5)y=0------(4)$$

Adding equation (3) and (4) we get,

$$[(D-\&)(D-5)+2]y=0$$

$$D^2-12D+37)y=0$$

This equation is similar to the second order ordinary linear differential equation.This can be solved by taking auxiliary equation which is,

$$m^2-12m+37=0$$

$$m=6\pm i$$

Since this cannot be solved directly we use the formula,

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The general solution for the given equation will be,

$$y=e^{6t}(c_1\cos t+c_2\sin t)-----(5)$$

Now $c_1 \,\&\, c_2$ are the arbitrary constants.Differentiating equation (5) with respect to ‘t’ we get,

$$D_y = 6e^{6t}\left[(c_1\cos t + c_2\sin t) + e^{6t}(-c_1\sin t + c_2\cos t)\right]$$

$$D_y = 6e^{6t}\left[(6c_1 + c_2)\cos t + (-c_1 + 6c_2)\sin t ------(6)\right]$$

Now substituting y and D_y in equation (2) we get the value of x,

$$x = \left(\frac{1}{2}\right) \times e^{6t}\left[(c_1 + c_2)\cos t + (-c_1 + c_2)\sin t ------(7)\right]$$

The equations (5) and (7) are the general solutions of the given simultaneous differential equation.

Method of differentiation:

Consider the following differential equations,

$$f_1(D)x + f_2(D)y = T_1 -------(1)$$

$$g_1(D)x + g_2(D)y = T_2 -------(2)$$

Sometimes x and y can be eliminated if we differentiate (1) and (2). For example assume that the given equations (1) and (2) relates four quantities $x, y, \frac{dx}{dt}, \frac{dy}{dt}$.

Differentiating (1) and (2) with respect to t we obtain four equations containing

$$x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}.$$

Eliminating three quantities,

$$y, \frac{dy}{dt}, \frac{d^2y}{dt^2}$$

From these four equations y is eliminated and we get an equation of the second order with x as the dependent and t as the independent variable. Solving this equation we get the value of x in terms of t.This is how the differentiation method can be used. Now substituting this value of 'x' in either equation (1) or (2) we get value of y in terms of t. This technique will be illustrated by the following example.

Example 2:

Determine the general solution for x and y for $\frac{dy}{dt} - y = t, \frac{dy}{dt} + x = 1.$

Solution:

$$Dx - y = t - - - - (1)$$

$$x + Dy = 1 - - - - (2)$$

Now differentiating equation (1) with respect to 't' will get,

$$D^2 x - Dy = 1 - - - - (3)$$

We can eliminate y by adding equation (3) and (2),

$$(D^2 + 1)x = 2 - - - - (4)$$

Then the auxiliary equation for the above equation will be,

$$m^2 + 1 = 0$$

$$C.F = c_1 \cos t + c_2 \sin t$$

This is the complementary function of the general solution for the above auxiliary equation.

$$P.I = \frac{1}{D^2 + 1} 2 = (1 + D^2)^{-1} 2$$

$$(1 + D^2)^{-1} = (1 - D^2 + ...)2 = 2$$

Hence the general solution x will be,

$$x = c_1 \cos t + c_2 \sin t + 2 ----(5)$$

From equation (1) we get,

$$y = c_2 \cos t - c_1 \sin t - t ----(6)$$

This is the solution for the given differential equation.

Summary:

In this lecture we have learned that the Simultaneous linear differential equations are

$$f_1(D)x + f_2(D)y = T_1 --------(1)$$

$$g_1(D)x + g_2(D)y = T_2 --------(2)$$

Where $T_1 \& T_2$ are functions of the independent variable 't' and $f_1(D), f_2(D), g_1(D), g_2(D)$ are all rational integral functions of D with constant co-efficient. And then the solution to this simultaneous linear differential equation is obtained by method of elimination and method of differentiation.

After listening to this lecture you can answer the following questions.

Questions:

1. What is known as simultaneous linear differential equation?
2. List different methods to find the solution of simultaneous linear differential equations.
3. List the step followed in finding solution for simultaneous linear differential equation by method of elimination.