Bachelor of Architecture

Mathematics

Lecture 11

In this lecture we are going to see the Linear differential equations (LDE), Second order LDE with constant co-efficient and the Summary.

Linear differential equations:

Linear differential equations are in which the dependent variable and its derivatives occurs only in the first degree and are not multiplied together. Thus the general linear differential equation of the nth order is of the form.

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots\ldots p_n y = x$$

Where $p_1, p_2 \ldots\ldots\ldots p_n$ and x functions of x only and above said equation can be reduced to second order as follows.

Keep n=2,

$$\frac{d^2 y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = x$$

This is the second order linear differential equation with constant co-efficient if $p_1 \;\&\; p_2$ are all constants.

Theorem on Solution:

If $y = y_1 \;\&\; y = y_2$ are the solution of linear differential equation then $c_1 y_1 + c_2 y_2 = u$ is also its solution.

The complete solution of differential equation has two parts,

i. Complementary function (C.F)
ii. Particular integral (P.I).

Operators D:

Denoting $\frac{d}{dx}, \frac{d^2}{dx^2}$ by $D \& D^2$ the differential equation can be written in symbolic form as,

$$D^2 + p_1 D + p_2)y = x$$

Thus the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity.

For example:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 3y = (D^2 + 2D - 3)y = (D+3)(D-1)y \text{ or}$$

$$(D-1)(D+3)y$$

Rules for finding the complementary functions:

Write the given differential equation in symbolic form,

$$i.e., \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

$$(D^2 + 2D - 3)y = 0$$

Equate the symbolic form to zero,

$$(D^2 + 2D - 3)y = 0$$

This is called auxiliary equations (A.E). Next this auxiliary equation has to be factorized. Then let $m_1 \& m_2$ be its roots.

Case I:

If $m_1 \& m_2$ are real and different then $y = c_1 e^{m1x} + c_2 e^{m2x}$ is the complementary function.

Case II:

If the two roots are equal $i.e., m_1 = m_2$ then $y = (c_1 + c_2)e^{m1x}$.

Case III:

If the roots are complex, $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$ then the complementary solution is $y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha + i\beta)x}$.

Example 1:

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

Solution:

Let us first write the differential equation in the operator form,

$$(D^2 + D - 2)y = 0$$

$$D^2 + D - 2 = 0$$

This equation is known as the auxiliary equation. So factorizing this equation we get,

$$(D + 2)(D - 1) = 0$$

$$D = -2, 1$$

$$y = c_1 e^{-2x} + c_2 e^{x}$$

This is the complete solution for the given differential equation.

Example 2:

Solve $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x=0$

Solution:

Here the x is a variable which is differentiated with respect to t. Now write the given equation in the operator mode.

$$(D^2+6D+9)x=0$$

Then the auxiliary form of this equation will be,

$$D^2+6D+9=0$$

Next factorizing this auxiliary form of equation we get,

$$(D+3)^2=0$$

$$D=-3,-3$$

Here the roots are equal hence the complementary function will be in the form,

$$x=(c_1+c_2t)e^{-3t}$$

Rules for finding the particular integral:

Consider the equation $(D^n+k_1D^{n-1}+k_2D^{n-2}+..........k_n)y=x$

The particular integral will be,

$$P.I.=\frac{1}{f(D)}e^{ax}\text{ put }D=a[f(a)\neq 0]$$

$$=x\frac{1}{f'(D)}e^{ax}\text{ put }D=a[f(a)=0,\ f'(a)\neq 0]$$

$= x^2 \dfrac{1}{f''(D)} e^{ax}$ put $D = a[f'(a) = 0,\ f''(a) \neq 0],$ and so on.

Case II:

When $x = \sin(ax+b)$ or $\cos(ax+b)$,

$$P.I = \frac{1}{\phi(D^2)} \sin(ax+b) \text{ or } \cos(ax+b)$$

Put $D^2 = -\alpha^2[\phi(-\alpha^2) \neq 0]$

$$= x \frac{1}{\phi'(D^2)} \sin(ax+b)$$

Put $D^2 = -\alpha^2[\phi(-\alpha^2) = 0, \phi'(-\alpha^2) \neq 0]$

$$= x^2 \frac{1}{\phi''(D^2)} \sin(ax+b)$$

Put $D^2 = -\alpha^2[\phi(-\alpha^2) = 0, \phi''(-\alpha^2) \neq 0]$ and so on.

Case III:

$$P.I = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

Expand $[f(D)]^{-1}$ in ascending powers of D by binomial theorem as for as D^m and operate on x^m term by term.

Case IV:

When $x = e^{ax} v$, where v is a function of x,

$P.I = \frac{1}{f(D)} e^{ax} v = e^{ax} \frac{1}{f(D+a)} v$ and then evaluate $\frac{1}{f(D+a)} v$ as in the cases I,II,III.

Case V:

When x is any function of x,

$$P.I = \frac{1}{f(D)} x$$

Resolve $\frac{1}{f(D)}$ into partial fraction and operate each partial fraction an x remembering that $\frac{1}{D-a} x = e^{ax} \int x e^{-ax} dx$.

Summary:

In this lecture we learned the Definition of linear differential equation is learnt. And then the solution of second order linear differential equation is learnt. Complementary function and particular integrals are the components of complete solution of linear differential equation.

After listening to this lecture you can answer the following questions.

Questions:

1. Define linear differential equations.
2. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$
3. Solve $(D^2 - 1)y = x, \sin 3x + \cos x$.