Bachelor of Architecture

Mathematics

Lecture 10

In this lecture we are going to discuss about the Maxima and Minima of a function. And then to understand this we need to undergo two steps called First derivative test and the second derivative test. And then we see the summary.

Maxima and Minima:

This is one of the important application of differentiation is finding out the maximum and minimum value of a function. Almost every function has some maximum and minimum value. We know that the algebraic function, trigonometric functions are all variable functions.

Point of Maxima and minima:

F(x) is said to have a maximum value at $x = a$ if f(a) is greater than any other value of f(x) in the neighborhood of x=a. Here the point x=a, is called point of maxima.

So if f(x) is maximum at x=a then $f(a) \geq f(x)$ for all x. Hence the point where a function attains its greatest or highest value is called point of maximum and the corresponding value of function is called maximum value.

Similarly the function f(x) is said to have minimum at x=a if f(a) is less than any of value of f(x) in neighborhood of x=a. Here the point x=a, is called point of minimum so if f(x) has minimum value at x=a. Then $f(a) \leq f(x)$ for all x $f(a) \leq f(x_i), x_i \in x$. Hence the point where a function attains its least value is called the point of maxima and the corresponding value of the function is called minimum value. The maximum and minimum values are called extreme values.

Conditions for maxima and minima values of a function:

The point where a function has a maximum or a minimum values are called turning point or critical points or stationary points. At these points tangents to the curve are always parallel to x-axis. So the necessary condition of maxima and minima is that $\frac{dy}{dx}=0$. So to obtain stationary points we put $\frac{dy}{dx}=0$.

First derivative test:

F(x) has maximum value at x=a, if f(x) ceases to increase at x=a, and begins to decrease when x>a. Thus in left neighborhood of x=a, f(x) increase $[f'(x) \rhd 0]$ and in right neighborhood of x=a, f(x) decreases to $[f'(x) \lhd 0]$.

Hence f(x) has maximum value at x=a, if f'(x) is > 0 if left neighborhood x=a and f'(x) is < 0 in right neighborhood of x=a. In other words y=f(x) maximum at x=a, if $\frac{dy}{dx}$ changes its sign form Positive to negative.

Similarly y=f(x) is minimum at x=a if $\frac{dy}{dx}$ changes its sign from negative to positive. Hence f(x) has maxima and minima at x=a if

I. $f'(x)=0,$ at x=a
II. If $\frac{dy}{dx}$ changes its sign from positive to negative in neighborhood of x=a then x=a, is point of maxima.
III. If $\frac{dy}{dx}$ changes its sign from negative to positive in neighborhood of x=a then x=a, is a point of minima.

Example 1:

Without using differentiation find maximum and minimum value of the function $(x-1)^2+4$.

Solution:

Let us take the function as y,

$y=(x-1)^2+4$

Where this $(x-1)^2$ is the perfect square which can never be negative, so

$$(x-1)^2 \geq 0$$

$$(x-1)^2 + 4 \geq 4$$

$$y \geq 4$$

Now if we take y=4 when x=1, so y is minimum at x=1. Hence the minimum value of y is 4 at x=1. These functions do not have any maximum value.

Second derivative test for maxima and minima:

We know that f(x) is maximum at x=a if f'(x) or $\frac{dy}{dx}$ changes its sign from positive to negative. Since f'(x) or $\frac{dy}{dx}$ itself is a function of x so change of sign of f'(x) from positive to negative implies that f'(x) or $\frac{dy}{dx}$ is a decreasing function. So that derivative of f'(x) or $\frac{dy}{dx}$ with respect to x should be negative.

$i.e., \frac{d}{dx}\left(\frac{dy}{dx}\right)_{x=a}$ is negative.

$$\left.\frac{d^2y}{dx^2}\right|_{x=a} \lhd 0$$

Thus f(x) is maximum at x=a if,

- $f'(x) = 0, at$ x=a
- $\left.\frac{d^2y}{dx^2}\right|_{x=a} \lhd 0$

In this similar way we can show that f(x) is minimum at x=a if,

- $f'(x) = 0, at$ x=a
- $\left.\frac{d^2y}{dx^2}\right|_{x=a} \rhd 0$

But there may be some points where $\frac{d^2y}{dx^2} = 0$ or there is no change in the sign of $\frac{dy}{dx}$ such points are called points of inflexion.

Example 2:

Determine the maximum and minimum value of $f(x) = 2x^3 - 15x^2 + 36x + 10$

Solution:

Let us assume the given derivative of function,

$f(x) = 2x^3 - 15x^2 + 36x + 10$

Finding the first and second derivate of the given function we get,

$f'(x) = 6x^2 - 30x + 36$

$f''(x) = 12x - 30$

Now equating the first derivative to zero we get,

$6x^2 - 30x + 36 = 0$

$x^2 - 5x + 6 = 0$

$x = 2,3$

Now we equate the second derivate of the given function to the x value.

$f''(x) = 12x - 30$

$f''(x)\big|_{x=2} = 12(2) - 30$

$= -6 \prec 0$

So x=2 is the point of maximum.Now substitute x=2 in the function then,

$f(2) = 2(2)^3 - 15(2)^2 + 36(2) + 10$ =38

$f''(x)\big|_{x=3} = 12(3) - 30 = 6 \succ 0$

$f(3) = 2(3)^3 - 15(3)^2 + 36(3) + 10 = 37$

This is the minimum value of the function and this minimum value is at the point x =3.

Summary:

In this lecture we learned that the important application of differentiations is finding out the maximum and minimum value of a function. F(X) is said to have a maximum value at x=a if f(x) sn the greater than any other values of f(x) in the neighborhood of x=a. This point is called point of maxima.

Similarly f(x) is said to have minimum at x=a if f(a) is less than any of the value of f(x) in neighborhood of x=a. First derivate test and second derivate test are the test exists for maxima and minima.

After listening to this lecture you can answer the following questions.

Questions:

1. Define maximum and minimum value of a function.
2. Find the maximum and minimum value of $f(x) = x^2.e^x$
3. What is known as second derivative test?